Extension of M-convexity and L-convexity to Polyhedral Convex Functions*

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The concepts of M-convex and L-convex functions were proposed by Murota in 1996 as two mutually conjugate classes of discrete functions over integer lattice points. M/L-convex functions are deeply connected with the well-solvability in nonlinear combinatorial optimization with integer variables. In this paper, we extend the concept of M-convexity and L-convexity to polyhedral convex functions, aiming at clarifying the well-behaved structure in well-solved nonlinear combinatorial optimization problems in real variables. The extended M/L-convexity often appear in nonlinear combinatorial optimization problems with piecewise-linear convex cost. We investigate the structure of polyhedral M-convex and L-convex functions from the dual viewpoint of analysis and combinatorics, and provide some properties and characterizations. It is also shown that polyhedral M/L-convex functions have nice conjugacy relationship.

Key Words: combinatorial optimization, matroid, base polyhedron, convex analysis, polyhedral convex function

1. INTRODUCTION

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In the area of combinatorial optimization, there exist many "well-solved" problems, i.e., the problems which have nice combinatorial structure and which can be solved efficiently (see [5, 23, 39]). Many researchers have been trying to identify the well-behaved structure in combinatorial optimization problems.

The concept of matroid, introduced by Whitney [47], plays an important role in the field of combinatorial optimization (see [46, 48, 49, 50]). Matroidal structure is closely related to the well-solvability of combinatorial optimization problems such as those on graphs and matroids, and can be found in fairly large number of efficiently solvable problems. Matroidal structure yields the tractability of problems in the following way:

- Global optimality is equivalent to local optimality, which implies the success of the so-called greedy algorithm for the problem of optimizing a linear function over a single matroid.
- A nice duality theorem, Edmonds' intersection theorem [9], guarantees the existence of a certificate for the optimality in the matroid intersection problem in terms of dual variables.

In 1970, Edmonds introduced the concept of polymatroid by extending that of matroid to sets of real vectors ([9], see also [46]). A polymatroid $P \subseteq \mathbf{R}_+^V$ is a polyhedron given as

$$P = \{x \in \mathbf{R}_+^V \mid \sum_{w \in V} x(w) \leq \rho(X) \ (\forall X \subseteq V)\}$$

by a submodular set function $\rho : 2^V \to \mathbf{R}$ with certain additional conditions, where $\mathbf{R}_+$ denotes the set of nonnegative reals, and ρ is called submodular if

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \qquad (\forall X, Y \subseteq V).$$

Polymatroids share nice combinatorial properties of matroids: for example, the greedy algorithm for matroids still works for polymatroids, and a duality holds for the polymatroid intersection problem. Fujishige, emphasizing the essential role of submodularity of ρ, generalized the concept of polymatroid to that of submodular system [16, 17].

In recent years, nonlinear combinatorial optimization problems are investigated more often due to theoretical interest and necessity in practical application. The nonlinear resource allocation problem (see [19, 22]) and the convex cost submodular flow problem (see [17, 21]) are examples of nonlinear combinatorial optimization problems. Both of the problems have nice combinatorial structures, which lead to efficient combinatorial algorithms. These results, however, do not completely fit in the framework of matroid, polymatroid, and submodular system.

The concepts of M-convex and L-convex functions, introduced by Murota [29, 30, 32], afford a nice framework for well-solved nonlinear combinatorial optimization problems. M-convex function is a natural extension of the concept of valuated matroid introduced by Dress–Wenzel [7, 8] (see also [26, 27, 28, 36]) as well as a quantitative generalization of the set of integral points in an integral base polyhedron [17]. L-convex function is an extension of submodular set function.

Let V be a finite set. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M-convex if $\mathrm{dom}_{\mathbf{Z}} f \neq \emptyset$ and it satisfies (M-EXC[**Z**]):

(M-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where

$$\begin{aligned} \mathrm{dom}_{\mathbf{Z}} f &= \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}, \\ \mathrm{supp}^+(x-y) &= \{w \in V \mid x(w) > y(w)\}, \\ \mathrm{supp}^-(x-y) &= \{w \in V \mid x(w) < y(w)\}, \end{aligned}$$

and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L-convex[1] if $\mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ and it satisfies (LF1[**Z**]) and (LF2[**Z**]):

(LF1[Z]) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}_{\mathbf{Z}} g)$,
(LF2[Z]) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r$ $(\forall p \in \mathrm{dom}_{\mathbf{Z}} g,\ \lambda \in \mathbf{Z})$,

where $p \wedge q, p \vee q$ $(\in \mathbf{R}^V)$ denote the vectors with $(p \wedge q)(v) = \min\{p(v), q(v)\}$, $(p \vee q)(v) = \max\{p(v), q(v)\}$ $(v \in V)$, and $\mathbf{1}$ $(\in \mathbf{R}^V)$ is the vector with each component being equal to one.

M/L-convex functions have nice properties:

- local optimality is equivalent to global optimality.
- M/L-convex functions can be extended to ordinary convex functions.
- M/L-convex functions are conjugate to each other.
- a (discrete) separation theorem and a Fenchel-type duality theorem hold for a pair of M-convex/M-concave (L-convex/L-concave) functions.

The minimization of M/L-convex functions can be done in polynomial time [11, 43]. Application of M-convex functions can be found in system analysis through polynomial matrices [25, 31, 33, 36], and in mathematical economics [6].

M-convexity and L-convexity appear in various nonlinear combinatorial optimization problems with integer variables. Such nice combinatorial

[1] In the original definition [32], an L-convex function is assumed to be integer-valued. See Remark 6.1.

properties, however, are enjoyed not only by combinatorial optimization problems in integer variables but also by those in real variables. We dwell on this point by considering the minimum cost flow/tension problems.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose we are given a family of piecewise-linear convex functions $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, each of which represents the cost of flow on the arc a. A function $\xi : A \to \mathbf{R}$ is called a flow. The boundary $\partial\xi : V \to \mathbf{R}$ of a flow ξ is given by

$$\begin{aligned} \partial\xi(v) = \sum\{\xi(a) \mid a(\in A) \text{ leaves } v\} \\ - \sum\{\xi(a) \mid a(\in A) \text{ enters } v\} \qquad (v \in V). \end{aligned} \tag{1}$$

Then, the cost function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ of the minimum cost flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is defined by

$$f(x) = \inf \left\{ \sum_{a \in A} f_a(\xi(a)) \;\middle|\; \begin{array}{l} \xi \in \mathbf{R}^A, \\ \partial\xi(w) = -x(w) \ (w \in T), \\ \partial\xi(w) = 0 \ (w \in V \setminus T) \end{array} \right\} \ (x \in \mathbf{R}^T). \tag{2}$$

Suppose we are given another family of piecewise-linear convex functions $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, each of which represents the cost of tension on the arc a. Any function $p : V \to \mathbf{R}$ is called a potential. Given a potential p, its coboundary $\delta p : A \to \mathbf{R}$ is defined by

$$\delta p(a) = p(u) - p(v) \qquad (a = (u, v) \in A). \tag{3}$$

Then, the cost function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ of the minimum cost tension that realizes a potential vector $p' \in \mathbf{R}^T$ is written as

$$g(p') = \inf \left\{ \sum_{a \in A} g_a(-\delta p(a)) \;\middle|\; \begin{array}{l} p \in \mathbf{R}^V, \\ p(w) = p'(w) \ (w \in T) \end{array} \right\} \ (p' \in \mathbf{R}^T). \tag{4}$$

It is well-known that the minimum cost flow/tension problems with piecewise-linear convex cost can be solved efficiently by various combinatorial algorithms (see [1, 42]). It can be shown that both f and g are polyhedral convex functions (see Section 4 for the definition of a polyhedral convex function), which is a direct extension of results in Iri [20] and Rockafellar [42] for the case of $|T| = 2$.

We consider here the cost functions $f_{\mathbf{Z}}$ and $g_{\mathbf{Z}}$ for the integer version of the minimum cost flow/tension problems:

$$\begin{aligned} f_{\mathbf{Z}}(x) &= \inf \left\{ \sum_{a \in A} f_a(\xi(a)) \;\middle|\; \begin{array}{l} \xi \in \mathbf{Z}^A, \\ \partial\xi(w) = -x(w) \ (w \in T), \\ \partial\xi(w) = 0 \ (w \in V \setminus T) \end{array} \right\} \quad (x \in \mathbf{Z}^T), \\ g_{\mathbf{Z}}(p') &= \inf \left\{ \sum_{a \in A} g_a(-\delta p(a)) \;\middle|\; \begin{array}{l} p \in \mathbf{Z}^V, \\ p(w) = p'(w) \ (w \in T) \end{array} \right\} \quad (p' \in \mathbf{Z}^T). \end{aligned}$$

It is shown in [32, 34, 35] that $f_{\mathbf{Z}}$ satisfies (M-EXC[**Z**]) and $g_{\mathbf{Z}}$ satisfies (LF1[**Z**]) and (LF2[**Z**]), i.e., $f_{\mathbf{Z}}$ and $g_{\mathbf{Z}}$ are M-convex and L-convex, respectively.

These results indicate that the polyhedral convex functions f and g defined by (2) and (4) must have nice combinatorial properties like M-convexity and L-convexity, respectively. As is shown later in Example 2.4, f satisfies the property (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \quad (0 \leq \forall \alpha \leq \alpha_0),$$

which is a generalization of (M-EXC[**Z**]), and g satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \quad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g, \forall \lambda \in \mathbf{R})$,

which can be obtained by generalizing (LF1[**Z**]) and (LF2[**Z**]), where

$$\begin{aligned} \mathrm{dom}\, f &= \{x \in \mathbf{R}^V \mid -\infty < f(x) < +\infty\}, \\ \mathrm{dom}\, g &= \{p \in \mathbf{R}^V \mid -\infty < g(p) < +\infty\}. \end{aligned}$$

The observation above indicates the possibility of extending the concepts of M-convexity and L-convexity to polyhedral convex functions. This can be done in the following way. For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, we call f M-convex if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies the property (M-EXC). Similarly, for a polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ we call g L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2).

The aim of this paper is to investigate the structures of polyhedral M-convex and L-convex functions from the dual viewpoint of analysis and combinatorics, and to provide a nice framework for well-solvable nonlinear combinatorial optimization problems in real variable. The organization of this paper is as follows.

Section 2 provides some natural classes of polyhedral M/L-convex functions. We also prove the polyhedral M-convexity and L-convexity of the functions f and g defined in (2) and (4).

To investigate polyhedral M/L-convex functions, we need to consider the set version of M/L-convexity. A polyhedron $B \subseteq \mathbf{R}^V$ is called M-convex if it is nonempty and satisfies (B-EXC):

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$x - \alpha(\chi_u - \chi_v) \in B,\ y + \alpha(\chi_u - \chi_v) \in B \qquad (0 \le \forall\alpha \le \alpha_0).$$

As is explained later in Theorem 3.3, an M-convex polyhedron is nothing but the base polyhedron of a submodular system [17]. Similarly, a polyhedron $D \subseteq \mathbf{R}^V$ is called L-convex if it is nonempty and satisfies (LS1) and (LS2):

(LS1) $p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D$,
(LS2) $p \in D \Longrightarrow p + \lambda\mathbf{1} \in D$ $(\forall\lambda \in \mathbf{R})$.

We investigate the structure of M/L-convex polyhedra in Section 3.

Section 4 shows fundamental properties of polyhedral M/L-convex functions. We present various equivalent axioms for polyhedral M/L-convex functions, and give some properties on local structure of polyhedral M/L-convex functions such as directional derivatives, subdifferentials, minimizers, etc. In Section 4, we also investigate positively homogeneous polyhedral M/L-convex functions, which are important subclasses of polyhedral M/L-convex functions. It is shown that positively homogeneous polyhedral M/L-convex functions have one-to-one correspondences with certain set functions, and also with L/M-convex polyhedra.

For a function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, its conjugate function $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$f^\bullet(p) = \sup_{x \in \mathbf{R}^V} \{\langle p, x\rangle - f(x)\} \qquad (p \in \mathbf{R}^V),$$

where $\langle p, x\rangle = \sum\{p(v)x(v) \mid v \in V\}$. It is shown in [32, 35] that there is a conjugacy relationship between integer-valued M/L-convex functions over the integer lattice. In Section 5, we show that the conjugacy relationship also exists for polyhedral M/L-convex functions. Section 5 also provides various characterization of polyhedral M/L-convex functions by local structures such as directional derivative, the set of minimizers, and subdifferentials.

As is mentioned above, the concepts of M/L-convexity were originally introduced for functions defined over the integer lattice [29, 30, 32]. In

Section 6, we clarify the relationship of M/L-convexity over the integer lattice and polyhedral M/L-convexity discussed in this paper.

Finally, some duality theorems on polyhedral M/L-convex functions are presented in Section 7. Although duality theorems in this section can be obtained by application of the existing theorems in convex analysis to polyhedral M/L-convex functions, they are worth stating in their own right in connection to integrality properties. We also explain a more general and stronger result on the transformation of polyhedral M/L-convex function called "network induction." The validity of network induction is proved by using the duality for polyhedral M/L-convex functions.

In this paper, we focus on polyhedral convex functions with M/L-convexity. The concepts of M/L-convexity can be further extended to general (not necessarily polyhedral) convex functions by considering the property (M-EXC) and properties (LF1), (LF2), respectively. Results in this direction will be reported elsewhere.

2. EXAMPLES OF POLYHEDRAL M/L-CONVEX FUNCTIONS

Polyhedral M-convex and L-convex functions have various examples. Let V be a nonempty finite set.

EXAMPLE 2.1 (affine functions). Let $B \subseteq \mathbf{R}^V$ be an M-convex polyhedron, $p_0 \in \mathbf{R}^V$, and $\alpha \in \mathbf{R}$. Then, the function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that

$$\operatorname{dom} f = B, \qquad f(x) = \langle p_0, x \rangle + \alpha \quad (x \in B)$$

is polyhedral M-convex with equality in (M-EXC), which is an immediate consequence of (B-EXC) for B.

Let $D \subseteq \mathbf{R}^V$ be an L-convex polyhedron, $x_0 \in \mathbf{R}^V$, and $\nu \in \mathbf{R}$. Then, the function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that

$$\operatorname{dom} g = D, \qquad g(p) = \langle p, x_0 \rangle + \nu \quad (p \in D)$$

is polyhedral L-convex with equality in the submodular inequality (LF1) and $r = \langle \mathbf{1}, x_0 \rangle$ in (LF2). ▮

We denote by $\mathcal{C}^1$ the class of one-dimensional piecewise-linear convex functions with a nonempty effective domain, i.e.,

$$\mathcal{C}^1 = \{\varphi \mid \varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}, \text{ piecewise-linear convex, } \operatorname{dom} \varphi \neq \emptyset\}. \quad (5)$$

A piecewise-linear convex function (in this paper) is nothing but a one-dimensional polyhedral convex function. Note that the effective domain $\operatorname{dom} \varphi$ of a function $\varphi \in \mathcal{C}^1$ is a closed interval on $\mathbf{R}$.

EXAMPLE 2.2. For $\varphi \in \mathcal{C}^1$, the function $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x_1, x_2) = \begin{cases} \varphi(x_1) & (x_1 + x_2 = 0), \\ +\infty & (\text{otherwise}) \end{cases}$$

is polyhedral M-convex. For $\psi \in \mathcal{C}^1$, the function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p_1, p_2) = \psi(p_1 - p_2) \quad ((p_1, p_2) \in \mathbf{R}^2)$$

is polyhedral L-convex. Furthermore, f and g are conjugate to each other if φ and ψ are conjugate to each other. ▮

EXAMPLE 2.3 (separable-convex functions). Let $B \subseteq \mathbf{R}^V$ be an M-convex polyhedron. For a family of functions $f_w \in \mathcal{C}^1$ $(w \in V)$, the function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} \displaystyle\sum_{w \in V} f_w(x(w)) & (x \in B), \\ +\infty & (x \notin B) \end{cases}$$

is polyhedral M-convex if $\mathrm{dom}\, f \neq \emptyset$.

Let $D \subseteq \mathbf{R}^V$ be an L-convex polyhedron. For functions $g_{uv} \in \mathcal{C}^1$ indexed by $u, v \in V$, the function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \begin{cases} \displaystyle\sum_{u,v \in V} g_{uv}(p(v) - p(u)) & (p \in D), \\ +\infty & (p \notin D) \end{cases}$$

is polyhedral L-convex with $r = 0$ in (LF2) if $\mathrm{dom}\, g \neq \emptyset$. ▮

EXAMPLE 2.4 (minimum cost flow/tension problems). In Introduction, we have defined the cost functions f and g of the minimum cost flow/tension problems by (2) and (4), respectively. We here state the following properties of f and g:

(i) f is polyhedral M-convex if $f(x_0)$ is finite for some $x_0 \in \mathbf{R}^T$.
(ii) g is polyhedral L-convex if $g(p_0)$ is finite for some $p_0 \in \mathbf{R}^T$.
(iii) Suppose that f_a and g_a are conjugate to each other for all $a \in A$. Then, f and g are conjugate to each other if one of the following conditions holds:

(a) $f(x_0)$ is finite (i.e., $-\infty < f(x_0) < +\infty$) for some $x_0 \in \mathbf{R}^T$,
(b) $g(p_0)$ is finite (i.e., $-\infty < g(p_0) < +\infty$) for some $p_0 \in \mathbf{R}^T$,

(c) $f(x_0) < +\infty$, $g(p_0) < +\infty$ for some $x_0 \in \mathbf{R}^T$, $p_0 \in \mathbf{R}^T$.

In the following, we prove (M-EXC) for f and (LF1), (LF2) for g. We omit the proofs of (iii) and the following properties:

- if $f(x_0)$ is finite for some $x_0 \in \mathbf{R}^T$, then $\mathrm{dom}\, f \neq \emptyset$ and $f > -\infty$,
- if $g(p_0)$ is finite for some $p_0 \in \mathbf{R}^T$, then $\mathrm{dom}\, g \neq \emptyset$ and $g > -\infty$,

since their proofs are just a straightforward extension of the results in [20, 42] for the case $|T| = 2$.

[(M-EXC) for f] Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x-y)$. Also, let $\xi, \eta \in \mathbf{R}^A$ be flows such that

$$\begin{aligned} \partial\xi(w) &= \begin{cases} -x(w) & (w \in T), \\ 0 & (w \in V \setminus T), \end{cases} & \partial\eta(w) &= \begin{cases} -y(w) & (w \in T), \\ 0 & (w \in V \setminus T), \end{cases} \\ f(x) &= \sum_{a \in A} f_a(\xi(a)), & f(y) &= \sum_{a \in A} f_a(\eta(a)). \end{aligned}$$

Note that the existence of such ξ and η is assured by the polyhedral convexity of each f_a. By a standard augmenting path argument we see that there exist $\pi : A \to \{0, \pm 1\}$ and $v \in \mathrm{supp}^-(x-y)$ $(\subseteq T)$ such that

$$\mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi - \eta), \quad \mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi - \eta), \quad \partial\pi = \chi_v - \chi_u.$$

Put

$$\alpha_0 = \min\{|\xi(a) - \eta(a)| \mid a \in A,\ \pi(a) \neq 0\} > 0.$$

Let α be any value with $0 \leq \alpha \leq \alpha_0$. Then, we have

$$\begin{aligned} &\partial(\xi - \alpha\pi) = -x + \alpha(\chi_u - \chi_v), \quad \partial(\eta + \alpha\pi) = -y - \alpha(\chi_u - \chi_v), \\ &f_a(\xi(a) - \alpha\pi(a)) + f_a(\eta(a) + \alpha\pi(a)) \leq f_a(\xi(a)) + f_a(\eta(a)) \quad (a \in A), \end{aligned}$$

which implies the inequality:

$$\begin{aligned} & f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \\ & \quad \leq \sum_{a \in A} \{f_a(\xi(a) - \alpha\pi(a)) + f_a(\eta(a) + \alpha\pi(a))\} \\ & \quad \leq \sum_{a \in A} \{f_a(\xi(a)) + f_a(\eta(a))\} \ = \ f(x) + f(y). \end{aligned}$$

Thus, f satisfies (M-EXC).

[(LF1) for g] Let $p', q' \in \mathrm{dom}\, g$, and $p, q \in \mathbf{R}^V$ be vectors with

$$\begin{aligned} & p(w) = p'(w) \ (w \in T), & & q(w) = q'(w) \ (w \in T), \\ & \sum_{a \in A} g_a(-\delta p(a)) = g(p'), & & \sum_{a \in A} g_a(-\delta q(a)) = g(q'). \end{aligned}$$

Note that the existence of such p and q is assured by the polyhedral convexity of each g_a. For $a = (u, v) \in A$, we have

$$g_a(-\delta(p \wedge q)(a)) + g_a(-\delta(p \vee q)(a)) \le g_a(-\delta p(a)) + g_a(-\delta q(a))$$

by the convexity of g_a. It is obvious that

$$(p \wedge q)(w) = (p' \wedge q')(w), \quad (p \vee q)(w) = (p' \vee q')(w) \quad (w \in T).$$

Hence, we have

$$\begin{aligned} g(p' \wedge q') + g(p' \vee q') &\le \sum_{a \in A} g_a(-\delta(p \wedge q)(a)) + \sum_{a \in A} g_a(-\delta(p \vee q)(a)) \\ &\le \sum_{a \in A} g_a(-\delta p(a)) + \sum_{a \in A} g_a(-\delta q(a)) = g(p') + g(q'). \end{aligned}$$

[(LF2) for g] Let $\mathbf{1}_T$ (resp. $\mathbf{1}_V$) be the vector in $\mathbf{R}^T$ (resp. $\mathbf{R}^V$) with each component being equal to one. For $p' \in \mathbf{R}^T$ and $\lambda \in \mathbf{R}$, we have

$$\begin{aligned} & g(p' + \lambda \mathbf{1}_T) \\ &= \inf \left\{ \sum_{a \in A} g_a(-\delta \tilde{p}(a)) \;\middle|\; \begin{array}{l} \tilde{p} \in \mathbf{R}^V, \\ \tilde{p}(w) = p'(w) + \lambda \ (w \in T) \end{array} \right\} \\ &= \inf \left\{ \sum_{a \in A} g_a(-\delta(p + \lambda \mathbf{1}_V)(a)) \;\middle|\; \begin{array}{l} p \in \mathbf{R}^V, \\ p(w) + \lambda = p'(w) + \lambda \ (w \in T) \end{array} \right\} \\ &= \inf \left\{ \sum_{a \in A} g_a(-\delta p(a)) \;\middle|\; \begin{array}{l} p \in \mathbf{R}^V, \\ p(w) = p'(w) \ (w \in T) \end{array} \right\} = g(p'). \quad \blacksquare \end{aligned}$$

3. M-CONVEX AND L-CONVEX POLYHEDRA

In this section, we introduce two classes of polyhedra, called M-convex polyhedra and L-convex polyhedra. In fact, M-convex and L-convex polyhedra are familiar objects in combinatorial optimization, and this section is a recapitulation of almost known facts on these polyhedra from our point of view. The results in this section will be the basis of argument on polyhedral M-convex and L-convex functions beginning in Section 4.

3.1. Definitions and Notation

We denote by $\mathbf{R}$ the set of reals, and by $\mathbf{Z}$ the set of integers. Also, denote by $\mathbf{R}_+$ the set of nonnegative reals. Throughout this paper, we assume that V is a nonempty finite set. For any finite set X, its cardinality is denoted by $|X|$. The characteristic vector of a subset $X \subseteq V$ is denoted by χ_X ($\in \{0,1\}^V$), i.e.,

$$\chi_X(w) = \begin{cases} 1 & (w \in X), \\ 0 & (w \in V \setminus X). \end{cases}$$

In particular, we use the notation $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_V$.

Let $x = (x(w) \mid w \in V) \in \mathbf{R}^V$. We define

$$\mathrm{supp}(x) = \{v \in V \mid x(v) \neq 0\},$$
$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\},$$
$$\|x\|_1 = \sum_{v \in V} |x(v)|, \qquad \|x\|_\infty = \max_{v \in V} |x(v)|,$$
$$\langle p, x \rangle = \sum_{v \in V} p(v)x(v) \ (p \in \mathbf{R}^V), \qquad x(X) = \sum_{v \in X} x(v) \ (X \subseteq V).$$

We sometimes write $\mathbf{0}_V$, $\mathbf{1}_V$, and $\langle p, x \rangle_V$ to indicate that $\mathbf{0}$, $\mathbf{1}$, p, and x are vectors in $\mathbf{R}^V$.

For $\varepsilon > 0$, we define

$$\mathrm{N}_\infty(x, \varepsilon) = \{y \in \mathbf{R}^V \mid \|y - x\|_\infty < \varepsilon\}.$$

For any $p, q \in \mathbf{R}^V$, $p \wedge q$ and $p \vee q$ denote the vectors in $\mathbf{R}^V$ such that

$$(p \wedge q)(w) = \min\{p(w), q(w)\}, \quad (p \vee q)(w) = \max\{p(w), q(w)\} \quad (w \in V).$$

For $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a(v) \leq b(v)$ $(v \in V)$, we define the interval $[a, b]$ ($\subseteq \mathbf{R}^V$) by

$$[a, b] = \{x \in \mathbf{R}^V \mid a(v) \leq x(v) \leq b(v) \ (v \in V)\}.$$

For any two vectors $p, q \in \mathbf{R}^V$, we denote $\mathrm{Box}[p, q] = [p \wedge q, p \vee q]$.

Let $S \subseteq \mathbf{R}^V$. The set S is called *convex* if $(1 - \alpha)x + \alpha y \in S$ holds for any $x, y \in S$ and any $\alpha \in [0, 1]$, and called *conic* if $\alpha x \in S$ holds for any $x \in S$ and any $\alpha > 0$. A conic set is also called a *cone*. The *convex hull* of S, denoted by $\mathrm{conv}(S)$, is the smallest convex set containing S; that is,

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \,\middle|\, k \geq 1, \ x_i \in S, \ \lambda_i \geq 0, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Note that S is convex if and only if $S = \mathrm{conv}(S)$. For any set $S \subseteq \mathbf{R}^V$, the *convex closure* of S, denoted by $\overline{S}$, is the smallest closed convex set containing S. Hence, $\overline{\mathrm{N}_\infty(x,\varepsilon)}$ denotes a closed neighborhood $\{y \in \mathbf{R}^V \mid \|y-x\|_\infty \leq \varepsilon\}$.

A set $S \subseteq \mathbf{R}^V$ is called *polyhedral* if it is represented as the intersection of a finite number of closed half-spaces, i.e., there exist some $\{p_i\}_{i=1}^k$ $(\subseteq \mathbf{R}^V)$ and $\{\alpha_i\}_{i=1}^k$ $(\subseteq \mathbf{R})$ $(k \geq 0)$ such that

$$S = \{x \in \mathbf{R}^V \mid \langle p_i, x\rangle \leq \alpha_i \ (\forall i = 1, \cdots, k)\}. \tag{6}$$

A polyhedral set is also called a *polyhedron*. Obviously, any polyhedron is a closed convex set.

We use the following convention involving $\pm\infty$:

$$\inf\{\alpha \mid \alpha \in \emptyset\} = +\infty, \quad \sup\{\alpha \mid \alpha \in \emptyset\} = -\infty, \quad +\infty \leq +\infty, \quad -\infty \leq -\infty,$$
$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0.$$

In this paper, we never use expressions such as $+\infty - \infty$ or $-\infty + \infty$.

3.2. M-convex Polyhedra

A polyhedron $B \subseteq \mathbf{R}^V$ is called *M-convex* if it is nonempty and satisfies (B-EXC):

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$x - \alpha(\chi_u - \chi_v) \in B, \quad y + \alpha(\chi_u - \chi_v) \in B \qquad (0 \leq \forall\alpha \leq \alpha_0).$$

We denote by $\mathcal{M}_0$ the family of M-convex polyhedra, i.e.,

$$\mathcal{M}_0 = \{B \subseteq \mathbf{R}^V \mid B : \text{ M-convex polyhedron}\}.$$

It is well-known as a folklore that what we call an "M-convex polyhedron" is nothing but the base polyhedron of a submodular system [17] (see also Theorem 3.3). We use the term "M-convex polyhedron" for denotational symmetry to "L-convex polyhedron."

3.2.1. Properties of M-convex Polyhedra

For any $S \subseteq \mathbf{R}^V$ and $x \in S$, the *exchange capacity* $\tilde{c}_S(x, v, u)$ $(u, v \in V)$ is defined by

$$\tilde{c}_S(x, v, u) = \sup\{\alpha \mid \alpha \in \mathbf{R}_+, \ x + \alpha(\chi_v - \chi_u) \in S\},$$

where the subscript S to $\tilde{c}$ may be omitted when there is no ambiguity. The property (B-EXC) can be rewritten in terms of exchange capacity, as follows:

(B-EXC′) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $\tilde{c}_B(x,v,u) > 0$, $\tilde{c}_B(y,u,v) > 0$.

We also consider a one-sided exchange property:

(B-EXC$_-$) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $\tilde{c}_B(x,v,u) > 0$.

The exchange property (B-EXC$_-$), apparently weaker than (B-EXC), is in fact equivalent to (B-EXC).

THEOREM 3.1. *For a polyhedron* $B \subseteq \mathbf{R}^V$, (B-EXC) $\iff$ (B-EXC$_-$).

Proof. Proof is given in Section 3.2.3. ▮

Fundamental properties of M-convex polyhedra are shown below. An M-convex polyhedron is contained in a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = r\}$ for some $r \in \mathbf{R}$.

THEOREM 3.2. *Let* $B \subseteq \mathbf{R}^V$ *be a polyhedron with* (B-EXC$_-$). *Then,* $x(V) = y(V)$ $(\forall x, y \in B)$.

Proof. Assume, to the contrary, that $x_0(V) > y_0(V)$ holds for some $x_0, y_0 \in B$. Let $x_* \in B$ be a vector with

$$\|x_* - y_0\|_1 = \inf\{\|x - y_0\|_1 \mid x \in B,\ x(V) = x_0(V)\}.$$

Note that $\mathrm{supp}^+(x_* - y_0) \neq \emptyset$. Let $u \in \mathrm{supp}^+(x_* - y_0)$. By (B-EXC$_-$), there exist some $v \in \mathrm{supp}^-(x_* - y_0)$ and a sufficiently small $\alpha > 0$ such that $x' = x_* - \alpha(\chi_u - \chi_v) \in B$. It, however, is a contradiction to the choice of x_* since $x'(V) = x_0(V)$ and $\|x' - y_0\|_1 = \|x_* - y_0\|_1 - 2\alpha$. ▮

We shall show that an M-convex polyhedron is described by a submodular set function. Let $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$ and denote $\mathrm{dom}\,\rho = \{X \subseteq V \mid \rho(X) < +\infty\}$. A function ρ is said to be *submodular* if it satisfies

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \quad (\forall X, Y \subseteq V). \tag{7}$$

We denote the class of (normalized) submodular set functions by

$$\mathcal{S} = \{\rho : 2^V \to \mathbf{R} \cup \{+\infty\} \mid \rho : \text{submodular},\ \rho(\emptyset) = 0,\ \rho(V) < +\infty\}.$$

A set function $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ is called *supermodular* if $-\mu$ is submodular. For any nonempty $B \subseteq \mathbf{R}^V$, define $\rho_B : 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho_B(X) = \sup_{x \in B} x(X) \quad (X \subseteq V). \tag{8}$$

For a set function $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, we define $\mathrm{B}(\rho) \subseteq \mathbf{R}^V$ by

$$\mathrm{B}(\rho) = \{x \in \mathbf{R}^V \mid x(X) \leq \rho(X) \ (X \subseteq V),\ x(V) = \rho(V)\}. \tag{9}$$

Note that the exchange capacity associated with $x \in \mathrm{B}(\rho)$ can be written as follows [17]:

$$\tilde{c}_{\mathrm{B}(\rho)}(x, v, u) = \min\{\rho(X) - x(X) \mid X \subseteq V,\ u \notin X,\ v \in X\} \quad (u, v \in V). \tag{10}$$

The following fact has been known to experts (cf. [4], [9], [46, Chapter 18]), but the precise statement cannot be found in the literature.

THEOREM 3.3.
(i) *For $B \in \mathcal{M}_0$, we have $\rho_B \in \mathcal{S}$ and $\mathrm{B}(\rho_B) = B$.*
(ii) *For $\rho \in \mathcal{S}$, we have $\mathrm{B}(\rho) \in \mathcal{M}_0$ and $\rho_{\mathrm{B}(\rho)} = \rho$.*
(iii) *The mappings $B \mapsto \rho_B$ $(B \in \mathcal{M}_0)$ and $\rho \mapsto \mathrm{B}(\rho)$ $(\rho \in \mathcal{S})$ provide one-to-one correspondences between $\mathcal{M}_0$ and $\mathcal{S}$, and are the inverse of each other.*

Proof. Proof is given later in Section 3.2.3. ▮

For any sets $S_1, S_2 \subseteq \mathbf{R}^V$, we denote by $S_1 + S_2$ $(\subseteq \mathbf{R}^V)$ the *Minkowski-sum* of S_1 and S_2, i.e.,

$$S_1 + S_2 = \{p_1 + p_2 \mid p_i \in S_i \ (i = 1, 2)\}.$$

THEOREM 3.4 ([17]). *For $B_1, B_2 \in \mathcal{M}_0$, the set $B_1 + B_2$ is M-convex with $B_1 + B_2 = \mathrm{B}(\rho_{B_1} + \rho_{B_2})$.*

REMARK 3.5. The intersection of two M-convex polyhedra is not necessarily an M-convex polyhedron, as shown in the example below.
Let $V = \{a, b, c, d\}$, and define $B_1, B_2 \subseteq \mathbf{R}^V$ as

$$B_1 = \mathrm{conv}\{(1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,0,1)\},$$
$$B_2 = \mathrm{conv}\{(1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,1,0), (1,0,0,1)\}.$$

Then, B_1 and B_2 are M-convex polyhedra, and the submodular functions $\rho_i = \rho_{B_i}$ $(i = 1, 2)$ defined by (8) are

$$\begin{aligned}
&\rho_i(\emptyset) = 0,\ \ \rho_i(\{v\}) = 1 \text{ for } v \in V\ \ (i = 1, 2),\\
&\rho_i(X) = 2 \text{ for } X \subseteq V \text{ with } |X| \geq 3\ (i = 1, 2),\\
&\rho_1(\{a, d\}) = \rho_1(\{b, c\}) = 1,\\
&\rho_1(\{a, b\}) = \rho_1(\{c, d\}) = \rho_1(\{a, c\}) = \rho_1(\{b, d\}) = 2,\\
&\rho_2(\{b, d\}) = 1,\\
&\rho_2(\{a, b\}) = \rho_2(\{a, c\}) = \rho_2(\{a, d\}) = \rho_2(\{b, c\}) = \rho_2(\{c, d\}) = 2.
\end{aligned}$$

We see that

$$B_1 \cap B_2 = \mathrm{conv}\{(1, 1, 0, 0), (1, 0, 1, 0), (0, 0, 1, 1)\},$$

which is not an M-convex polyhedron. Note that the associated set function $\rho = \rho_{B_1 \cap B_2}$ in (8) is

$$\begin{aligned}
&\rho(\{a, d\}) = \rho(\{b, c\}) = \rho(\{b, d\}) = 1,\\
&\rho(\{a, b\}) = \rho(\{c, d\}) = \rho(\{a, c\}) = 2,\\
&\rho(X) = \rho_1(X) = \rho_2(X) \text{ if } |X| \neq 2,
\end{aligned}$$

which is not submodular. ▮

Separation theorems of the following form hold for submodular functions and M-convex polyhedra. The latter half of Theorem 3.7 is already shown in [34, Theorem 3.6].

THEOREM 3.6 (Frank [12]). *Let $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ be a pair of functions with $\rho, -\mu \in \mathcal{S}$. If $\mu(X) \leq \rho(X)$ $(\forall X \subseteq V)$, then there exists $x_* \in \mathbf{R}^V$ such that*

$$\mu(X) \leq x_*(X) \leq \rho(X) \qquad (\forall X \subseteq V).$$

Moreover, if ρ and μ are integer-valued, then there exists such $x_* \in \mathbf{Z}^V$.

THEOREM 3.7. *Let $B_1, B_2 \in \mathcal{M}_0$. If $B_1 \cap B_2 = \emptyset$, then there exists $p_* \in \{0, 1\}^V \cup \{-\chi_V\}$ such that*

$$\inf_{x \in B_1} \langle p_*, x\rangle - \sup_{x \in B_2} \langle p_*, x\rangle > 0. \tag{11}$$

Moreover, if both of B_1 and B_2 are integral polyhedra, i.e., $B_i = \overline{B_i \cap \mathbf{Z}^V}$ $(i = 1, 2)$, then "> 0" in (11) can be replaced by "≥ 1."

Proof. Put $\rho_i = \rho_{B_i}$ $(i = 1, 2)$. If $\rho_1(V) \neq \rho_2(V)$, then we can choose $p_* = \chi_V$ or $-\chi_V$. Hence, we may assume that $\rho_1(V) = \rho_2(V)$. Define the functions ρ and μ by

$$\mu(X) = \rho_1(V) - \rho_1(V \setminus X),\ \rho(X) = \rho_2(X) \qquad (X \subseteq V).$$

Then, $\rho, -\mu \in \mathcal{S}$ and there is no $x \in \mathbf{R}^V$ with $\mu(X) \leq x(X) \leq \rho(X)$ $(\forall X \subseteq V)$. By Theorem 3.6, there exists some $X_0 \subseteq V$ such that $\mu(X_0) > \rho(X_0)$. Since

$$\inf_{x \in B_1} \langle \chi_{X_0}, x \rangle = \mu(X_0) > \sup_{x \in B_2} \langle \chi_{X_0}, x \rangle = \rho(X_0),$$

we can choose $p_* = \chi_{X_0}$. The second claim follows from the integrality of B_1, B_2. ▌

3.2.2. *M♮-convex polyhedra*

Theorem 3.2 shows that an M-convex polyhedron is contained in a hyperplane of the form $\{x \in \mathbf{R}^V \mid x(V) = r\}$ for some $r \in \mathbf{R}$. Therefore, any M-convex polyhedron loses no information by the projection onto a $(|V|-1)$-dimensional space. We call a polyhedron $Q \subseteq \mathbf{R}^V$ *M♮-convex* if the set $\tilde{Q}$ $(\subseteq \mathbf{R}^{\tilde{V}})$ defined by

$$\tilde{Q} = \{(x_0, x) \in \mathbf{R}^{\tilde{V}} \mid x \in Q,\ x_0 = -x(V)\} \tag{12}$$

is an M-convex polyhedron, where $\tilde{V} = \{v_0\} \cup V$. In fact, by the results of [15, 17], what we name M♮-convex polyhedron here is nothing but a generalized polymatroid in [13, 14], where a *generalized polymatroid* is a nonempty set $Q \subseteq \mathbf{R}^V$ given by

$$Q = \{x \in \mathbf{R}^V \mid \mu(X) \leq x(X) \leq \rho(X)\ (X \subseteq V)\} \tag{13}$$

with a pair of submodular/supermodular functions $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ satisfying the following inequality:

$$\rho(X) - \mu(Y) \geq \rho(X \setminus Y) - \mu(Y \setminus X) \quad (X, Y \subseteq V).$$

M♮-convex polyhedron is an essentially equivalent concept to M-convex polyhedron, while the class of M♮-convex polyhedra properly contains that of M-convex polyhedra. Every property of M-convex polyhedra can be restated in terms of M♮-convex polyhedra, and vice versa.

M$^\natural$-convex polyhedra can be characterized by various exchange properties, as follows (cf. [37, Remark 5.2], [45]). For $Q \subseteq \mathbf{R}^V$ and $u, v \in V$, we denote

$$\begin{aligned}\tilde{c}_Q(x, \emptyset, u) &= \sup\{\alpha \mid \alpha \in \mathbf{R}_+,\ x - \alpha\chi_u \in Q\},\\ \tilde{c}_Q(x, v, \emptyset) &= \sup\{\alpha \mid \alpha \in \mathbf{R}_+,\ x + \alpha\chi_v \in Q\}.\end{aligned}$$

(G-EXC) $\forall x, y \in Q$, $\forall u \in \mathrm{supp}^+(x-y)$, either (i) or (ii) (or both) holds:
(i) $\exists v \in \mathrm{supp}^-(x-y)$ such that $\tilde{c}_Q(x, v, u) > 0$ and $\tilde{c}_Q(y, u, v) > 0$,
(ii) $\tilde{c}_Q(x, \emptyset, u) > 0$ and $\tilde{c}_Q(y, u, \emptyset) > 0$.

(G-EXC$_-$) $\forall x, y \in Q$, $\forall u \in \mathrm{supp}^+(x-y)$, either $\tilde{c}_Q(x, v, u) > 0$ ($\exists v \in \mathrm{supp}^-(x-y)$) or $\tilde{c}_Q(x, \emptyset, u) > 0$.

THEOREM 3.8. *Let $Q \subseteq \mathbf{R}^V$ be a nonempty polyhedron. Then,*

$$Q \text{ is M}^\natural\text{-convex} \iff Q \text{ satisfies (G-EXC)} \iff Q \text{ satisfies (G-EXC}_-\text{)}.$$

Proof. Proof is given later in Section 3.2.3. ▮

The following property, which seems more natural when described in terms of M$^\natural$-convex polyhedron, is obvious from (13).

THEOREM 3.9. *Let $Q \subseteq \mathbf{R}^V$ be an M$^\natural$-convex polyhedron. Then, $[x, y] \subseteq Q$ holds for any $x, y \in Q$ with $x \leq y$.*

3.2.3. Proofs

This section provides the proofs of Theorems 3.1, 3.3, and 3.8.

We use the following results on submodular set functions. A family $\mathcal{D} \subseteq 2^V$ is called a *distributive lattice* if $X \cap Y, X \cup Y \in \mathcal{D}$ holds for any $X, Y \in \mathcal{D}$. Note that $\mathrm{dom}\,\rho$ of a submodular function $\rho \in \mathcal{S}$ is a distributive lattice.

THEOREM 3.10. *Let $\rho \in \mathcal{S}$. For any $x \in \mathrm{B}(\rho)$, the family $\mathcal{D}(x) = \{X \subseteq V \mid x(X) = \rho(X)\}$ is a distributive lattice with $\{\emptyset, V\} \subseteq \mathcal{D}(x)$.*

We first prove a slightly stronger claim than Theorem 3.3 (i). A sequence $\{X_i\}_{i=0}^k$ ($k \geq 0$) of distinct elements in 2^V with $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k$ is called a *chain*.

LEMMA 3.11. *Let $B \subseteq \mathbf{R}^V$ be a nonempty polyhedron with* (B-EXC$_-$), *and $\{X_i\}_{i=0}^k$ $(k \geq 0)$ be a chain such that $\{X_i\}_{i=0}^k \subseteq \mathrm{dom}\,\rho_B$. Then, there exists $x_* \in B$ with $x_*(X_i) = \rho_B(X_i)$ $(i = 0, 1, \cdots, k)$.*

Proof. The claim is shown by induction on the integer k. Let $x_k \in B$ be a vector with $x_k(X_k) = \rho_B(X_k)$. By the inductive hypothesis, there exists a vector $x \in B$ satisfying $x(X_i) = \rho_B(X_i)$ $(i = 0, 1, \cdots, k-1)$. We may assume that x minimizes the value $\|x - x_k\|_1$ of all such vectors. Suppose that $x(X_k) < \rho_B(X_k)$. By (B-EXC$_-$), there exist $u \in \mathrm{supp}^+(x - x_k) \setminus X_k$ and $v \in \mathrm{supp}^-(x - x_k)$ such that $x' = x - \alpha(\chi_u - \chi_v) \in B$ with a sufficiently small $\alpha > 0$. Here $v \in V \setminus X_{k-1}$ holds since $u \in V \setminus X_{k-1}$ and $x'(X_{k-1}) \leq \rho_B(X_{k-1}) = x(X_{k-1})$. The vector x' satisfies $x'(X_i) = \rho_B(X_i)$ $(i = 0, 1, \cdots, k-1)$ and $\|x' - x_k\|_1 = \|x - x_k\|_1 - 2\alpha$, a contradiction. Hence $x(X_k) = \rho_B(X_k)$. ▮

For any polyhedron $S \subseteq \mathbf{R}^V$, a vector $x \in S$ is called an *extreme point* of S if there are no $y_1, y_2 \in S \setminus \{x\}$ and $\alpha \in \mathbf{R}$ with $0 < \alpha < 1$ such that $x = \alpha y_1 + (1 - \alpha) y_2$.

THEOREM 3.12 ([17, Theorem 3.22]). *Let $\rho \in \mathcal{S}$. Then, $x \in \mathbf{R}^V$ is an extreme point of* $\mathrm{B}(\rho)$ *if and only if there exists a chain $\{X_i\}_{i=0}^{|V|}$ such that*

$$\begin{array}{l} \{X_i\}_{i=0}^{|V|} \subseteq \mathrm{dom}\,\rho, \quad X_0 = \emptyset, \quad X_{|V|} = V, \\ x(X_i) = \rho(X_i) \quad (i = 0, 1, \cdots, |V|). \end{array}$$

LEMMA 3.13. *For any nonempty polyhedron $B \subseteq \mathbf{R}^V$ with* (B-EXC$_-$), *we have $\rho_B \in \mathcal{S}$ and $B = \mathrm{B}(\rho_B)$.*

Proof. The outline of the proof is similar to that for [4, Lemma 5.2].

We first consider the case when B is bounded. Let $X, Y \in \mathrm{dom}\,\rho_B$. From Lemma 3.11, there exists $x_* \in B$ with $x_*(X \cap Y) = \rho_B(X \cap Y)$ and $x_*(X \cup Y) = \rho_B(X \cup Y)$, which implies the inequality

$$\begin{aligned} \rho_B(X) + \rho_B(Y) &\geq x_*(X) + x_*(Y) \\ &= x_*(X \cap Y) + x_*(X \cup Y) \ = \ \rho_B(X \cap Y) + \rho_B(X \cup Y). \end{aligned}$$

Therefore, $\rho_B \in \mathcal{S}$. Lemma 3.11 and Theorem 3.12 imply that any extreme point of $\mathrm{B}(\rho_B)$ is contained in B, which shows that $\mathrm{B}(\rho_B) \subseteq B$. The reverse inclusion is obvious.

Next, assume that B is unbounded. Let x_0 be any vector in B. For each nonnegative integer k, put $B_k = \{x \in B \mid \|x - x_0\|_\infty \leq k\}$ and $\rho_k = \rho_{B_k}$. Since each B_k is a bounded M-convex polyhedron, it holds that $\rho_k \in \mathcal{S}$ and $B_k = \mathrm{B}(\rho_k)$. Hence, we have $\mathrm{B}(\rho_B) = \bigcup_{k=1}^{\infty} \mathrm{B}(\rho_k) = \bigcup_{k=1}^{\infty} B_k = B$, and

$$\begin{aligned}\rho_B(X) + \rho_B(Y) &= \lim_{k\to\infty} \{\rho_k(X) + \rho_k(Y)\} \\ &\geq \lim_{k\to\infty} \{\rho_k(X \cap Y) + \rho_k(X \cup Y)\} \\ &= \rho_B(X \cap Y) + \rho_B(X \cup Y)\end{aligned}$$

for any $X, Y \in \mathrm{dom}\,\rho_B$, i.e., $\rho_B \in \mathcal{S}$. ▮

Next, we prove Theorem 3.3 (ii).

LEMMA 3.14 (cf. [17]). *Let $\rho \in \mathcal{S}$, $x \in \mathrm{B}(\rho)$, and $u \in V$. Put*

$$X_0 = \{v \in V \mid \tilde{c}(x, v, u) = 0\}, \quad Y_0 = \{v \in V \mid \tilde{c}(x, u, v) > 0\}.$$

Then, we have $u \in V \setminus X_0$, $x(X_0) = \rho(X_0)$, and $u \in Y_0$, $x(Y_0) = \rho(Y_0)$.

Note that the set Y_0 in Lemma 3.14 is nothing but the dependence function $\mathrm{dep}(x, u)$ defined in [17].

We consider another one-sided exchange property:

(B-EXC$_+$) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that $\tilde{c}_B(y, u, v) > 0$.

LEMMA 3.15. *For any $\rho \in \mathcal{S}$, the set* $\mathrm{B}(\rho)$ *satisfies* (B-EXC$_-$) *and* (B-EXC$_+$).

Proof. We prove (B-EXC$_-$) only, since (B-EXC$_+$) can be shown in the similar way. Let $x, y \in \mathrm{B}(\rho)$ with $x \neq y$ and $u \in \mathrm{supp}^+(x - y)$. To the contrary assume $\tilde{c}(x, v, u) = 0$ for all $v \in \mathrm{supp}^-(x - y)$. Put $X_0 = \{v \in V \mid \tilde{c}(x, v, u) = 0\}$. By Lemma 3.14 we have $x(X_0) = \rho(X_0) \geq y(X_0)$, whereas $x(X_0) < y(X_0)$ since $u \in V \setminus X_0$ and $\mathrm{supp}^-(x - y) \subseteq X_0$, a contradiction. ▮

LEMMA 3.16. *Let $B \subseteq \mathbf{R}^V$ be a nonempty polyhedron with* (B-EXC$_-$) *and* (B-EXC$_+$). *For any $x, y \in B$, and $v \in \mathrm{supp}^-(x - y)$, there exists $y' \in B$ such that*

$$\begin{aligned}&y'(v) = x(v), \quad x(w) \geq y'(w) \geq y(w) \quad (w \in \mathrm{supp}^+(x - y)),\\ &y'(w) = y(w) \quad (w \notin \{v\} \cup \mathrm{supp}^+(x - y)).\end{aligned}$$

Proof. Put $\mathrm{supp}^+(x-y) = \{u_1, u_2, \cdots, u_k\}$ $(k \geq 1)$, and $y_0 = y$. For $i = 1, 2, \cdots, k$, we define $\alpha_i \in \mathbf{R}_+$ and $y_i \in B$ iteratively as follows:

$$\alpha_i = \min\{y_{i-1}(v) - x(v),\ x(u_i) - y_{i-1}(u_i),\ \tilde{c}(y_{i-1}, u_i, v)\},$$
$$y_i = y_{i-1} + \alpha_i(\chi_{u_i} - \chi_v).$$

Assume $y_k(v) > x(v)$. By (B-EXC$_-$), there exists some $u_i \in \mathrm{supp}^-(y_k - x) \subseteq \mathrm{supp}^+(x-y)$ such that $\hat{y} = y_k - \alpha(\chi_v - \chi_{u_i}) \in B$ for a sufficiently small $\alpha > 0$. Since $u_i \in \mathrm{supp}^+(\hat{y} - y_i)$ and $\mathrm{supp}^-(\hat{y} - y_i) = \{v\}$, (B-EXC$_+$) implies $y_i + \beta(\chi_{u_i} - \chi_v) \in B$ for a sufficiently small $\beta > 0$, a contradiction to the fact that $\alpha_i = \tilde{c}(y_{i-1}, u_i, v)$. ▮

LEMMA 3.17. *For any $\rho \in \mathcal{S}$, the set* $\mathrm{B}(\rho)$ *satisfies* (B-EXC).

Proof. Firstly note that $\mathrm{B}(\rho)$ satisfies (B-EXC$_-$) and (B-EXC$_+$) by Lemma 3.15. Let $x, y \in \mathrm{B}(\rho)$ with $x \neq y$ and $u \in \mathrm{supp}^+(x-y)$. Put $Y_0 = \{v \in V \mid \tilde{c}(y, u, v) > 0\}$. Applying Lemma 3.16 repeatedly, we have some $y' \in \mathrm{B}(\rho)$ such that

$$y'(v) = x(v) \ \ (v \in \mathrm{supp}^-(x-y) \setminus Y_0),$$
$$x(w) \geq y'(w) \geq y(w) \ \ (w \in \mathrm{supp}^+(x-y)),$$
$$y'(w) = y(w) \ \ (w \notin (\mathrm{supp}^-(x-y) \setminus Y_0) \cup \mathrm{supp}^+(x-y)).$$

Moreover, $y'(w) = y(w)$ holds for $w \in \mathrm{supp}^+(x-y) \cap Y_0$ since otherwise $y'(Y_0) > y(Y_0) = \rho(Y_0)$ by Lemma 3.14, a contradiction. In particular, $y'(u) = y(u) < x(u)$. Applying (B-EXC$_-$) to x, y', and $u \in \mathrm{supp}^+(x - y')$, we have $v_0 \in \mathrm{supp}^-(x-y')$ with $\tilde{c}(x, v_0, u) > 0$. From $\mathrm{supp}^-(x-y') = \mathrm{supp}^-(x-y) \cap Y_0$ follows $\tilde{c}(y, u, v_0) > 0$. ▮

THEOREM 3.18 ([17]). *For any $\rho \in \mathcal{S}$ we have* $\mathrm{B}(\rho) \neq \emptyset$ *and* $\rho_{B(\rho)} = \rho$.

Theorem 3.3 (iii) is an immediate corollary of Lemmas 3.13, 3.17, and Theorem 3.18.

We then give the proof of Theorem 3.1.

Proof of Theorem 3.1 The implication (B-EXC) $\Rightarrow$ (B-EXC$_-$) is obvious. Let $B \subseteq \mathbf{R}^V$ be a nonempty polyhedron with (B-EXC$_-$). Then, we have $\rho_B \in \mathcal{S}$ and $B = \mathrm{B}(\rho_B)$, as shown in Lemma 3.13. Hence, Lemma 3.17 implies (B-EXC) for B. ▮

Finally, we prove Theorem 3.8. Since the implications "Q is M$^\natural$-convex $\Rightarrow$ (G-EXC) $\Rightarrow$ (G-EXC$_-$)" are easy to see, we show "(G-EXC$_-$) $\Rightarrow$ Q is M$^\natural$-convex" only.

We first note that (G-EXC$_-$) for Q is equivalent to the following property for $\tilde{Q}$ defined by (12):

(G$^\sharp$-EXC$_-$) $\forall x, y \in \tilde{Q}$, $\forall u \in \mathrm{supp}^+(x-y) \setminus \{v_0\}$, $\exists v \in \mathrm{supp}^-(x-y) \cup \{v_0\}$ such that $\tilde{c}_{\tilde{Q}}(x, v, u) > 0$.

Using this property, we prove the submodularity of $\rho_{\tilde{Q}}$ and the equation $\tilde{Q} = \mathrm{B}(\rho_{\tilde{Q}})$, where $\rho_{\tilde{Q}} : 2^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ and $\mathrm{B}(\rho_{\tilde{Q}}) \subseteq \mathbf{R}^{\tilde{V}}$ are defined by (8) and (9), respectively. The proof is similar to that of Theorem 3.3 (i).

LEMMA 3.19. *Let $\{X_i\}_{i=0}^k$ $(k \geq 0)$ be a chain such that $\{X_i\}_{i=0}^k \subseteq \mathrm{dom}\, \rho_{\tilde{Q}}$. Then, there exists $x_* \in \tilde{Q}$ with $x_*(X_i) = \rho_{\tilde{Q}}(X_i)$ $(i = 0, 1, \cdots, k)$.*

Proof. The claim is shown by induction on the integer k. By the inductive hypothesis and the definition of $\rho_{\tilde{Q}}$, there exist vectors $x, y \in \tilde{Q}$ such that

$$x(X_i) = \rho_{\tilde{Q}}(X_i) \qquad (i = 1, 2, \cdots, k), \tag{14}$$

$$y(X_0) = \rho_{\tilde{Q}}(X_0). \tag{15}$$

Let $x_*, y_* \in \tilde{Q}$ be a pair of vectors minimizing the value $||x_* - y_*||_1$ among all pairs of vectors $x, y \in \tilde{Q}$ satisfying (14), (15), and $x(X_0) = \max\{x'(X_k) \mid x' \in \tilde{Q},\ (14)\}$. We further assume that x_* and y_* maximize the value $y_*(v_0) - x_*(v_0)$ among all such vectors.

Claim $y_*(w) \leq x_*(w) \qquad (\forall w \in V \setminus (X_0 \cup \{v_0\}))$.

Assume, to the contrary, that there exists some $w_0 \in V \setminus (X_0 \cup \{v_0\})$ with $y_*(w_0) > x_*(w_0)$. Then, (G$^\sharp$-EXC$_-$) implies either (a-1), (a-2) or (a-3):

(a-1) $\exists v \in X_0$ such that $\tilde{c}_{\tilde{Q}}(y_*, v, w_0) > 0$,
(a-2) $\exists v \in \mathrm{supp}^-(y_* - x_*) \setminus X_0$ such that $\tilde{c}_{\tilde{Q}}(y_*, v, w_0) > 0$,
(a-3) $v_0 \in V \setminus X_0$, $y_*(v_0) \geq x_*(v_0)$, and $\tilde{c}_{\tilde{Q}}(y_*, v, w_0) > 0$ for $v = v_0$.

Put $y' = y_* - \alpha(\chi_{w_0} - \chi_v) \in \tilde{Q}$, where α is a sufficiently small positive number. In either case we have a contradiction:

$$\begin{array}{lll}
\text{(a-1)} & \Longrightarrow & y'(X_0) > \rho_{\tilde{Q}}(X_0), \\
\text{(a-2)} & \Longrightarrow & y'(X_0) = \rho_{\tilde{Q}}(X_0),\ ||y' - x_*||_1 < ||y_* - x_*||_1, \\
\text{(a-3)} & \Longrightarrow & y'(X_0) = \rho_{\tilde{Q}}(X_0),\ ||y' - x_*||_1 = ||y_* - x_*||_1, \\
 & & y'(v_0) - x_*(v_0) > y_*(v_0) - x_*(v_0).
\end{array}$$

[End of the proof of Claim]

Suppose that either $v_0 \in V \setminus X_0$ and $x_*(v_0) \geq y_*(v_0)$, or $v_0 \in X_0$ holds. Then, for $i = 1, 2, \cdots, k$, we have

$$\begin{aligned} y_*(X_i) &= y_*(\tilde{V}) - y_*(\tilde{V} \setminus X_i) \\ &\geq x_*(\tilde{V}) - x_*(\tilde{V} \setminus X_i) \ = \ x_*(X_i) \ = \ \rho_{\tilde{Q}}(X_i), \end{aligned}$$

implying $y_*(X_i) = \rho_{\tilde{Q}}(X_i)$. It remains to consider the case where

$$v_0 \in V \setminus X_0, \qquad x_*(v_0) < y_*(v_0). \tag{16}$$

Assume, to the contrary, that $x_*(X_0) < \rho_{\tilde{Q}}(X_0) = y_*(X_0)$. Then, there exists some $u \in \mathrm{supp}^+(x_* - y_*) \setminus X_0$. Hence, (G$^\sharp$-EXC$_-$) implies either (b-1) or (b-2):

(b-1) $\exists v \in \mathrm{supp}^-(x_* - y_*) \cap X_0$ such that $\tilde{c}_{\tilde{Q}}(x_*, v, u) > 0$,
(b-2) $\tilde{c}_{\tilde{Q}}(x_*, v, u) > 0$ for $v = v_0$.

Put $x' = x_* + \alpha(\chi_v - \chi_u) \in \tilde{Q}$, where α is a sufficiently small positive number. In case of (b-1), we have $u \in X_1 \setminus X_0$ since x_* satisfies (14). Therefore, x' also satisfies (14) and $x'(X_0) > x_*(X_0)$, a contradiction. In case of (b-2), we have $x'(X_0) = x_*(X_0)$ and $||x' - y_*||_1 < ||x_* - y_*||_1$ by (16), a contradiction. ▮

In the same way as the proof of Lemma 3.13, we can show that $\rho_{\tilde{Q}} \in \mathcal{S}$ and $\tilde{Q} = \mathrm{B}(\rho_{\tilde{Q}})$ by using Lemma 3.19 and Theorem 3.12.

3.3. L-convex Polyhedra

A polyhedron $D \subseteq \mathbf{R}^V$ is called *L-convex* if it is nonempty and satisfies (LS1) and (LS2):

(LS1) $p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D$,
(LS2) $p \in D \Longrightarrow p + \lambda \mathbf{1} \in D\ (\forall \lambda \in \mathbf{R})$.

We denote by $\mathcal{L}_0$ the family of L-convex polyhedra, i.e.,

$$\mathcal{L}_0 = \{D \subseteq \mathbf{R}^V \mid D: \text{ L-convex polyhedron}\}.$$

3.3.1. Properties of L-convex Polyhedra

The properties (LS1) and (LS2) can be characterized by local properties:

($\mathbf{LS1_{loc}}$-a) $\forall p \in D$, $\exists \varepsilon > 0$ such that

$$p_1, p_2 \in D \cap \overline{\mathrm{N}_\infty(p,\varepsilon)} \Longrightarrow p_1 \wedge p_2,\ p_1 \vee p_2 \in D,$$

($\mathbf{LS1_{loc}}$-b) $\forall p, q \in D$, $0 < \exists \varepsilon \leq 1$ such that

$$(1-\varepsilon)p + \varepsilon(p \wedge q) \in D, \quad (1-\varepsilon)p + \varepsilon(p \vee q) \in D,$$

($\mathbf{LS2_{loc}}$) $\forall p \in D$, $\exists \varepsilon > 0$ such that $p + \lambda \mathbf{1} \in D$ $(\forall \lambda \in [0, \varepsilon])$.

THEOREM 3.20. *Let $D \subseteq \mathbf{R}^V$ be a closed set.*
(i) *If D is convex, then* (LS1) $\Longleftrightarrow$ ($\mathrm{LS1_{loc}}$-a) $\Longleftrightarrow$ ($\mathrm{LS1_{loc}}$-b).
(ii) (LS2) $\Longleftrightarrow$ ($\mathrm{LS2_{loc}}$).

Proof. (i): Since (LS1) $\Rightarrow$ ($\mathrm{LS1_{loc}}$-a) is obvious, we show ($\mathrm{LS1_{loc}}$-a) $\Rightarrow$ ($\mathrm{LS1_{loc}}$-b) $\Rightarrow$ (LS1).

Assume ($\mathrm{LS1_{loc}}$-a). Let $p, q \in D$, and ε be the value in ($\mathrm{LS1_{loc}}$-a) associated with p. The convexity of D implies $(1-\varepsilon')p + \varepsilon' q \in D \cap \overline{\mathrm{N}_\infty(p,\varepsilon)}$ for a sufficiently small $\varepsilon' > 0$. Hence, we have

$$\begin{aligned} p \wedge \{(1-\varepsilon')p + \varepsilon' q\} &= (1-\varepsilon')p + \varepsilon'(p \wedge q) \in D, \\ p \vee \{(1-\varepsilon')p + \varepsilon' q\} &= (1-\varepsilon')p + \varepsilon'(p \vee q) \in D, \end{aligned}$$

i.e., ($\mathrm{LS1_{loc}}$-b) holds.

Next assume ($\mathrm{LS1_{loc}}$-b) for D to prove (LS1). Let $p, q \in D$. We show $p \wedge q \in D$ only, since $p \vee q \in D$ can be shown similarly. Set

$$\begin{aligned} \alpha_* &= \sup\{\alpha \mid 0 \leq \alpha \leq 1,\ (1-\alpha)p + \alpha(p \wedge q) \in D\}, \\ p_* &= (1-\alpha_*)p + \alpha_*(p \wedge q)\ (\in D). \end{aligned}$$

Assume $\alpha_* < 1$. By ($\mathrm{LS1_{loc}}$-b), there exists some ε with $0 < \varepsilon \leq 1$ such that

$$(1-\varepsilon)p_* + \varepsilon(p_* \wedge q) = (1 - \varepsilon - \alpha_* + \varepsilon\alpha_*)p + (\varepsilon + \alpha_* - \varepsilon\alpha_*)(p \wedge q) \in D,$$

a contradiction to the definition of α_*. Hence $\alpha_* = 1$ and $p \wedge q \in D$ hold.

(ii): We show the direction ($\mathrm{LS2_{loc}}$) $\Longrightarrow$ (LS2) only. Let $p \in D$ and $\lambda_* = \sup\{\lambda \mid \lambda \in \mathbf{R}_+,\ p + \lambda\mathbf{1} \in D\}$. To the contrary assume that $\lambda_* < +\infty$. Then, we have $p_* = p + \lambda_*\mathbf{1} \in D$. However, ($\mathrm{LS2_{loc}}$) implies $p_* + \varepsilon\mathbf{1} \in D$ for some $\varepsilon > 0$, a contradiction to the definition of λ_*. Thus we have $p + \lambda\mathbf{1} \in D$ for all $\lambda \geq 0$. Similarly, we have $p + \lambda\mathbf{1} \in D$ for all $\lambda \leq 0$. ▌

We show the system of inequalities which describes the polyhedral structure of L-convex polyhedra.

A function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ with $\gamma(v,v) = 0$ ($\forall v \in V$) is called a *distance function*. For a distance function γ, define the set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^V$ by

$$\mathrm{D}(\gamma) = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq \gamma(u,v) \ (u,v \in V)\}. \tag{17}$$

Given a nonempty set $D \subseteq \mathbf{R}^V$, the function $\gamma_D : V \times V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$\gamma_D(u,v) = \sup_{p \in D} \{p(v) - p(u)\}. \tag{18}$$

Note that γ_D is a distance function, and $D \subseteq \mathrm{D}(\gamma_D)$ holds in general.

Given a distance function γ, we denote by G_γ the directed graph with the vertex set V and the arc set $A_\gamma = \{(u,v) \mid u,v \in V,\ u \neq v,\ \gamma(u,v) < +\infty\}$, where the value $\gamma(u,v)$ is regarded as the length of the arc $(u,v) \in A_\gamma$. For $u,v \in V$, let $\overline{\gamma}(u,v)$ be the length of a shortest path from u to v in G_γ, i.e.,

$$\overline{\gamma}(u,v) = \inf\{ \sum_{(u',v') \in P} \gamma(u',v') \mid P \subseteq A_\gamma,\ P: \text{ path from } u \text{ to } v\}.$$

It is well-known that $\overline{\gamma} > -\infty$ if and only if G_γ has no negative cycle.

We consider the *triangle inequality*

$$\gamma(v_1,v_2) + \gamma(v_2,v_3) \geq \gamma(v_1,v_3) \quad (\forall v_1,v_2,v_3 \in V) \tag{19}$$

for a distance function, and let $\mathcal{T}$ be the family of distance functions with triangle inequality, i.e.,

$$\mathcal{T} = \{\gamma \mid \gamma : V \times V \to \mathbf{R} \cup \{+\infty\},\ \gamma(v,v) = 0 \ (v \in V),\ \gamma \text{ satisfies (19)}\}.$$

Note that for any distance function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$, if $\overline{\gamma} > -\infty$ then $\overline{\gamma} \in \mathcal{T}$, and if $\gamma \in \mathcal{T}$ then $\overline{\gamma} = \gamma$.

LEMMA 3.21. *Let $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ be a distance function.*
(i) $\mathrm{D}(\gamma) \neq \emptyset \iff$ *G_γ has no negative cycle.*
(ii) $\mathrm{D}(\gamma) \neq \emptyset \Longrightarrow \mathrm{D}(\gamma) = \mathrm{D}(\overline{\gamma})$.
(iii) $\mathrm{D}(\gamma) \neq \emptyset \Longrightarrow \mathrm{D}(\gamma) \in \mathcal{L}_0$.
(iv) *If $\gamma \in \mathcal{T}$, then $\mathrm{D}(\gamma) \neq \emptyset$ and $\gamma = \gamma_{\mathrm{D}(\gamma)}$.*
(v) *If $\gamma(u,v) \in \mathbf{Z} \cup \{+\infty\}$ ($\forall u,v \in V$), then $\mathrm{D}(\gamma)$ is an integral polyhedron, i.e., $\mathrm{D}(\gamma) = \overline{\mathrm{D}(\gamma) \cap \mathbf{Z}^V}$.*

Proof. (i), (ii): These are well-known facts. See, e.g., [20, 23].

(iii): (LS2) for $\mathrm{D}(\gamma)$ is obvious. To show (LS1) for $\mathrm{D}(\gamma)$, we take $p, q \in \mathrm{D}(\gamma)$. Then, it is easy show that

$$(p \wedge q)(v) - (p \wedge q)(u) \leq \gamma(u,v), \quad (p \vee q)(v) - (p \vee q)(u) \leq \gamma(u,v)$$

for any $u, v \in V$, i.e., $p \wedge q, p \vee q \in D$.

(iv): Since $\gamma \in \mathcal{T}$, the graph G_γ has no negative cycle and therefore $\mathrm{D}(\gamma) \neq \emptyset$ by (i). We have

$$\gamma(u,v) = \overline{\gamma}(u,v) = \sup_{p \in \mathrm{D}(\gamma)} \{p(v) - p(u)\} = \gamma_{\mathrm{D}(\gamma)}(u,v),$$

where the second equality is by a fundamental result concerning the shortest path problem.

(v): The claim is obvious from the total unimodularity of the coefficient matrix of the system of inequalities in (17). ▮

LEMMA 3.22. *For any nonempty polyhedron $D \subseteq \mathbf{R}^V$, we have $\gamma_D \in \mathcal{T}$. Moreover, if $D \in \mathcal{L}_0$, then $\mathrm{D}(\gamma_D) = D$.*

Proof. The first claim is clear from the following inequality: for any $v_1, v_2, v_3 \in V$, we have

$$\begin{aligned}\gamma_D(v_1,v_2) + \gamma_D(v_2,v_3) &= \sup_{p \in D}\{p(v_2) - p(v_1)\} + \sup_{p \in D}\{p(v_3) - p(v_2)\} \\ &\geq \sup_{p \in D}\{p(v_3) - p(v_1)\} = \gamma_D(v_1,v_3).\end{aligned}$$

Suppose $D \in \mathcal{L}_0$. We show that $\mathrm{D}(\gamma_D) \subseteq D$. Let $q \in \mathrm{D}(\gamma_D)$. For any $u, v \in V$ there exists $p_{uv} \in D$ with $p_{uv}(v) - p_{uv}(u) \geq q(v) - q(u)$, where we may assume that $p_{uv}(u) = q(u)$ and $p_{uv}(v) \geq q(v)$ by (LS2). For each $u \in V$, the vector $p_u = \bigvee_{v \in V} p_{uv}$ ($\in D$) satisfies $p_u(u) = q(u)$, $p_u(v) \geq q(v)$ ($\forall v \in V$). Therefore, $q = \bigwedge_{u \in V} p_u \in D$. ▮

THEOREM 3.23.
(i) *For $D \in \mathcal{L}_0$, we have $\gamma_D \in \mathcal{T}$ and $\mathrm{D}(\gamma_D) = D$.*
(ii) *For $\gamma \in \mathcal{T}$, we have $\mathrm{D}(\gamma) \in \mathcal{L}_0$ and $\gamma_{\mathrm{D}(\gamma)} = \gamma$.*
(iii) *The mappings $D \mapsto \gamma_D$ ($D \in \mathcal{L}_0$) and $\gamma \mapsto \mathrm{D}(\gamma)$ ($\gamma \in \mathcal{T}$) provide a one-to-one correspondence between $\mathcal{L}_0$ and $\mathcal{T}$, and are the inverse of each other.*

Proof. Immediate from Lemma 3.21 (iii), (iv), and Lemma 3.22. ▌

As a special case of L-convex polyhedra, we investigate the class of *L-convex cones*, which are conic L-convex polyhedra. This amounts to restricting the function values of γ to $\{0, +\infty\}$. We show below the relationship of L-convex cones with "transitive" directed graphs, and with distributive lattices.

A directed graph $G = (V, A)$ is called *transitive* if for any $(u, v), (v, w) \in A$ we have $(u, w) \in A$. Given a set of vectors $D \subseteq \mathbf{R}^V$, we define the directed graph $G_D = (V, A_D)$ by

$$A_D = \{(u, v) \mid u, v \in V,\ p(u) \geq p(v)\ (\forall p \in D)\}.$$

For a directed graph $G = (V, A)$, the set D_G $(\subseteq \mathbf{R}^V)$ is defined by

$$D_G = \{p \in \mathbf{R}^V \mid p(u) \geq p(v)\ (\forall (u, v) \in A)\}.$$

The next theorem is an immediate consequence of Theorem 3.23.

THEOREM 3.24.
(i) *For any L-convex cone $D \subseteq \mathbf{R}^V$, $G_D = (V, A_D)$ is transitive and $D_{G_D} = D$.*
(ii) *For any transitive directed graph $G = (V, A)$, D_G $(\subseteq \mathbf{R}^V)$ is an L-convex cone and $G_{D_G} = G$.*
(iii) *The mappings $D \mapsto G_D$ and $G \mapsto D_G$ provide a one-to-one correspondence between L-convex cones D $(\subseteq \mathbf{R}^V)$ and transitive directed graphs $G = (V, A)$, and are the inverse of each other.*

On the other hand, Birkhoff's representation theorem [2] yields a one-to-one correspondence between transitive directed graphs $G = (V, A)$ and distributive lattices $\mathcal{F}$ $(\subseteq 2^V)$ with $\{\emptyset, V\} \subseteq \mathcal{F}$. This fact, together with Theorem 3.24, provides a one-to-one correspondence between L-convex cones and distributive lattices, as follows.

Given a set of vectors $D \subseteq \mathbf{R}^V$, we define $\mathcal{F}_D \subseteq 2^V$ by

$$\mathcal{F}_D = \{X \subseteq V \mid \chi_X \in D\}.$$

For any family of subsets $\mathcal{F} \subseteq 2^V$ with $\{\emptyset, V\} \subseteq \mathcal{F}$, define $D_{\mathcal{F}} \subseteq \mathbf{R}^V$ by

$$D_{\mathcal{F}} = \{\sum_{X \in \mathcal{F}} \lambda_X \chi_X \mid \lambda_X \geq 0\ (X \in \mathcal{F} \setminus \{V\})\}.$$

THEOREM 3.25.
(i) *For an L-convex cone* $D \subseteq \mathbf{R}^V$, $\mathcal{F}_D$ *is a distributive lattice with* $\{\emptyset, V\} \subseteq \mathcal{F}_D$, *and* $D_{\mathcal{F}_D} = D$.
(ii) *For a distributive lattice* $\mathcal{F} \subseteq 2^V$ *with* $\{\emptyset, V\} \subseteq \mathcal{F}$, $D_{\mathcal{F}}$ *is an L-convex cone and* $\mathcal{F}_{D_{\mathcal{F}}} = \mathcal{F}$. *Moreover, for any* $p \in D_{\mathcal{F}}$ *there exists a chain* $\mathcal{F}_0$ *with* $\{\emptyset, V\} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}$ *and* $\lambda_X \in \mathbf{R}$ $(X \in \mathcal{F}_0)$ *such that* $\lambda_X \geq 0$ $(\forall X \in \mathcal{F}_0 \setminus \{V\})$ *and* $p = \sum_{X \in \mathcal{F}_0} \lambda_X \chi_X$.
(iii) *The mappings* $D \mapsto \mathcal{F}_D$ *and* $\mathcal{F} \mapsto D_{\mathcal{F}}$ *provide a one-to-one correspondence between L-convex cones* D $(\subseteq \mathbf{R}^V)$ *and distributive lattices* $\mathcal{F}$ $(\subseteq 2^V)$ *with* $\{\emptyset, V\} \subseteq \mathcal{F}$, *and are the inverse of each other.*

The intersection of L-convex polyhedra is also L-convex.

THEOREM 3.26. *Let* $D_1, D_2 \in \mathcal{L}_0$. *Define* $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ *by* $\gamma(u, v) = \min\{\gamma_{D_1}(u, v), \gamma_{D_2}(u, v)\}$ $(u, v \in V)$. *Then,*
(i) $D_1 \cap D_2 = \mathrm{D}(\gamma)$, (ii) $D_1 \cap D_2 \neq \emptyset \iff G_\gamma$ *has no negative cycle,*
(iii) $D_1 \cap D_2 \neq \emptyset \Longrightarrow D_1 \cap D_2 \in \mathcal{L}_0$.

Proof. The claim (i) follows from $D_1 \cap D_2 = \mathrm{D}(\gamma_{D_1}) \cap \mathrm{D}(\gamma_{D_2}) = \mathrm{D}(\gamma)$. The claims (ii) and (iii) are by (i) and (iii) of Lemma 3.21, respectively. ▮

REMARK 3.27. The Minkowski-sum of two L-convex polyhedra is not necessarily an L-convex polyhedron, as shown below.
Put $V = \{a, b, c, d\}$. Let $\gamma_1, \gamma_2 : V \times V \to \{0, 1\}$ be such that

$$\gamma_1(u, v) = 1 \iff (u, v) \in \{(c, a), (c, b), (d, a), (d, b), (d, c)\},$$
$$\gamma_2(u, v) = 1 \iff (u, v) \in \{(b, a), (b, c), (d, a), (d, b), (d, c)\}.$$

Since $\gamma_1, \gamma_2 \in \mathcal{T}$, we have $\mathrm{D}(\gamma_1), \mathrm{D}(\gamma_2) \in \mathcal{L}_0$ by Lemma 3.21. However, $D = \mathrm{D}(\gamma_1) + \mathrm{D}(\gamma_2)$ is not L-convex. Let

$$p_1 = (1, 1, 0, 0),\ p_2 = q_1 = (0, 0, 0, 0),\ q_2 = (1, 0, 1, 0).$$

Then, it holds that $p_i, q_i \in \mathrm{D}(\gamma_i)$ $(i = 1, 2)$. Hence, we have $p_1 + p_2, q_1 + q_2 \in D$, but $(p_1 + p_2) \wedge (q_1 + q_2) = (1, 0, 0, 0) \notin D$. Note that

$$\{p \in \mathrm{D}(\gamma_i) \mid p(d) = 0\} = \mathrm{conv}(S_i)\ (i = 1, 2),$$
$$\{p \in D \mid p(d) = 0\} = \mathrm{conv}(S_1 + S_2),$$

where

$$S_1 = \{(0, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0)\},$$
$$S_2 = \{(0, 0, 0, 0), (1, 0, 1, 0), (1, 1, 1, 0)\}.$$ ▮

Separation theorems of the following form hold for distance functions with triangle inequality and for L-convex polyhedra.

THEOREM 3.28. *Let* $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$, $\nu : V \times V \to \mathbf{R} \cup \{-\infty\}$ *be functions such that* $\gamma, -\nu \in \mathcal{T}$. *Suppose that*

$$\sum_{i=1}^{k} \nu(v_{2i+1}, v_{2i}) \leq \sum_{i=1}^{k} \gamma(v_{2i-1}, v_{2i})$$

holds for any distinct elements $\{v_i\}_{i=1}^{2k} \subseteq V$ *with* $v_{2k+1} = v_1$. *Then, there exists* $p_* \in \mathbf{R}^V$ *with*

$$\nu(u,v) \leq p_*(v) - p_*(u) \leq \gamma(u,v) \qquad (\forall u, v \in V). \tag{20}$$

Moreover, if γ *and* ν *are integer-valued, then there exists such* $p_* \in \mathbf{Z}^V$.

Proof. The existence of $p_* \in \mathbf{R}^V$ with (20) is equivalent to $\mathrm{D}(\nu^\sharp) \cap \mathrm{D}(\gamma) \neq \emptyset$, where $\nu^\sharp : V \times V \to \mathbf{R} \cup \{+\infty\}$ is defined by $\nu^\sharp(u,v) = -\nu(v,u)$ $(u, v \in V)$. Define $\eta : V \times V \to \mathbf{R} \cup \{+\infty\}$ by $\eta(u,v) = \min\{\gamma(u,v), \nu^\sharp(u,v)\}$ $(u,v \in V)$. By Theorem 3.26 (ii), it suffices to show that G_η has no negative cycle. Suppose, to the contrary, that $\{v_1, v_2, \cdots, v_h\} \subseteq V$ $(h \geq 1)$ forms a negative cycle. Since γ and $\nu^\sharp$ satisfy the triangle inequality, we may assume that $h = 2k$ $(k \geq 1)$ and

$$\eta(v_{2i-1}, v_{2i}) = \gamma(v_{2i-1}, v_{2i}),\ \eta(v_{2i}, v_{2i+1}) = -\nu(v_{2i+1}, v_{2i})\ (i = 1, 2, \cdots, k),$$

where $v_{2k+1} = v_1$. This implies $\sum_{i=1}^{k} \nu(v_{2i+1}, v_{2i}) > \sum_{i=1}^{k} \gamma(v_{2i-1}, v_{2i})$, a contradiction. The integrality assertion follows from the argument above and Lemma 3.21 (v). ▮

REMARK 3.29. The assumption in Theorem 3.28 cannot be replaced by the weaker condition

$$\nu(u,v) \leq \gamma(u,v) \qquad (\forall u, v \in V),$$

as shown in the following example.

Put $V = \{a, b, c, d\}$, and define $\gamma : V \times V \to \mathbf{R}$, $\nu : V \times V \to \mathbf{R}$ as follow:

γ	a	b	c	d
a	0	0	1	−1
b	2	0	1	1
c	1	−1	0	0
d	1	1	2	0

ν	a	b	c	d
a	0	0	−1	−1
b	0	0	−1	−1
c	−1	−1	0	0
d	−1	−1	0	0

where it reads as $\gamma(a,d) = -1$, $\gamma(b,a) = 2$, etc. We can easily check that $\gamma, -\nu \in \mathcal{T}$ and $\nu(u,v) \leq \gamma(u,v)$ $(\forall u, v \in V)$. However, there is no $p_* \in \mathbf{R}^V$ satisfying (20). Suppose, to the contrary, that such $p_* \in \mathbf{R}^V$ exists. Then, we have

$$\begin{aligned}
p_*(a) - p_*(b) &\leq \min\{\gamma(b,a), -\nu(a,b)\} = 0, \\
p_*(b) - p_*(c) &\leq \min\{\gamma(c,b), -\nu(b,c)\} = -1, \\
p_*(c) - p_*(d) &\leq \min\{\gamma(d,c), -\nu(c,d)\} = 0, \\
p_*(d) - p_*(a) &\leq \min\{\gamma(a,d), -\nu(d,a)\} = -1.
\end{aligned}$$

Addition of these inequalities yields $0 \leq -2$, a contradiction. Note that the assumption of Theorem 3.28 is not satisfied since $\nu(a,b) + \nu(c,d) = 0 > -2 = \gamma(c,b) + \gamma(a,d)$. ▮

The latter part of the following theorem is already shown in [34, Theorem 5.6].

THEOREM 3.30. *Let $D_1, D_2 \in \mathcal{L}_0$. If $D_1 \cap D_2 = \emptyset$, then there exists $x_* \in \{0, \pm 1\}^V$ such that*

$$\inf_{p \in D_1} \langle p, x_* \rangle - \sup_{p \in D_2} \langle p, x_* \rangle > 0. \tag{21}$$

Moreover, if D_1 and D_2 are integral polyhedra, i.e., $D_i = \overline{D_i \cap \mathbf{Z}^V}$ $(i = 1,2)$, then "> 0" in (21) can be replaced by "≥ 1".

Proof. Put $\gamma_i = \gamma_{D_i}$ $(i = 1,2)$. Since $\mathrm{D}(\gamma_1) \cap \mathrm{D}(\gamma_2) = \emptyset$, it follows from Theorem 3.28 that there exists a sequence of distinct elements $\{v_i\}_{i=1}^{2k} \subseteq V$ $(k \geq 1)$ such that $\sum_{i=1}^{k} \gamma_1(v_{2i-1}, v_{2i}) + \sum_{i=1}^{k} \gamma_2(v_{2i}, v_{2i+1}) < 0$, where $v_{2k+1} = v_1$. Let $x_* \in \{0, \pm 1\}^V$ be a vector such that $x_*(v_{2i}) = -1$, $x_*(v_{2i-1}) = 1$ $(i = 1, \cdots, k)$, and $x_*(w) = 0$ otherwise. Then, we have

$$\begin{aligned}
&\inf_{p \in D_1} \langle p, x_* \rangle - \sup_{p \in D_2} \langle p, x_* \rangle \\
&= \inf_{p \in D_1} \sum_{i=1}^{k} \{p(v_{2i-1}) - p(v_{2i})\} - \sup_{p \in D_2} \sum_{i=1}^{k} \{p(v_{2i+1}) - p(v_{2i})\} \\
&\geq -\sum_{i=1}^{k} \gamma_1(v_{2i-1}, v_{2i}) - \sum_{i=1}^{k} \gamma_2(v_{2i}, v_{2i+1}) \ > \ 0.
\end{aligned}$$

The integrality assertion is obvious from the inequality above. ▮

3.3.2. $L^\natural$-convex polyhedra

Due to the property (LS2), an L-convex polyhedron loses no information when restricted to a hyperplane $\{p \in \mathbf{R}^V \mid p(v) = 0\}$ for any $v \in V$. We call a polyhedron $P \subseteq \mathbf{R}^V$ *$L^\natural$-convex* if the set $\tilde{P} \subseteq \mathbf{R}^{\tilde{V}}$ defined by

$$\tilde{P} = \{(\lambda, p + \lambda\mathbf{1}) \mid p \in P,\ \lambda \in \mathbf{R}\}$$

is an L-convex polyhedron, where $\tilde{V} = \{v_0\} \cup V$. We see that L$^\natural$-convex polyhedra are essentially the same as L-convex polyhedra, while the class of L$^\natural$-convex polyhedra properly contains that of L-convex polyhedra.

Intervals are the only polyhedra that are both L$^\natural$- and M$^\natural$-convex.

THEOREM 3.31 (cf. [38, Lemma 5.7]). *A polyhedron $S \subseteq \mathbf{R}^V$ is both polyhedral $M^\natural$-convex and polyhedral $L^\natural$-convex if and only if S is an interval, i.e., it is represented as $S = [a, b]$ for some $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a(v) \leq b(v)$ $(v \in V)$.*

Proof. Consider the polyhedral description of S in terms of linear inequalities of the form $\langle p, x\rangle \leq \alpha$. For an L$^\natural$-convex set, $p = \chi_v - \chi_u$ or $p = \pm\chi_v$ for some elements $u, v \in V$ by Theorem 3.23 (i), while for M$^\natural$-convex set, $p = \pm\chi_X$ for some subset $X \subseteq V$ by (13). Therefore, S is an interval. ▮

REMARK 3.32. There exists no polyhedron which is both M-convex and L-convex, i.e., $\mathcal{M}_0 \cap \mathcal{L}_0 = \emptyset$ (Proof: If $S \in \mathcal{M}_0 \cap \mathcal{L}_0$, then Theorem 3.2 implies $x(V) = y(V)$ for any $x, y \in S$, whereas (LS2) implies that $x + \lambda\mathbf{1} \in S$ for any $\lambda \in \mathbf{R}$.) ▮

4. POLYHEDRAL M-CONVEX AND L-CONVEX FUNCTIONS

4.1. Review of Fundamental Results on Polyhedral Convex Functions

This section is devoted to a summary of the relevant results on polyhedral convex functions. See [41, 44] for more accounts.

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ be a function. The *epigraph* of f, denoted by epi$f$, is the set

$$\mathrm{epi}f = \{(x, \alpha) \mid x \in \mathbf{R}^V,\ \alpha \in \mathbf{R},\ \alpha \geq f(x)\}\ (\subseteq \mathbf{R}^V \times \mathbf{R}).$$

The effective domain of f, denoted by $\mathrm{dom}\, f$, is the set

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^V \mid -\infty < f(x) < +\infty\}.$$

We call f *convex* if $\mathrm{epi} f$ is a convex set in $\mathbf{R}^V \times \mathbf{R}$. A function f is called *concave* if $-f$ is convex. When $f > -\infty$, f is convex if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad (\forall x, y \in \mathbf{R}^V,\ \forall \alpha \in [0,1]).$$

A convex function is said to be *polyhedral* if its epigraph is polyhedral.

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$. The *(convex) conjugate* $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of f is defined by

$$f^\bullet(p) = \sup_{x \in \mathbf{R}^V} \{\langle p, x\rangle - f(x)\} \qquad (p \in \mathbf{R}^V). \tag{22}$$

THEOREM 4.1 ([41, Theorem 19.2]).
(i) *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, *its conjugate function* $f^\bullet$ *is also polyhedral convex with* $\mathrm{dom}\, f^\bullet \neq \emptyset$ *and* $f^\bullet > -\infty$, *and* $f^{\bullet\bullet} = f$.
(ii) *The mapping* $f \mapsto f^\bullet$ *induces a symmetric one-to-one correspondence in the class of polyhedral convex functions* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$.

A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called *positively homogeneous* if $f(\alpha x) = \alpha f(x)$ for any $x \in \mathbf{R}^V$ and $\alpha > 0$. Note that $f(\mathbf{0}) = 0$ if $\mathrm{dom}\, f \neq \emptyset$.

THEOREM 4.2 ([41, Corollary 4.7.1]). *If* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *is a positively homogeneous convex function, then* $f(\sum_{i=1}^k \alpha_i x_i) \leq \sum_{i=1}^k \alpha_i f(x_i)$ *for any* $x_i \in \mathbf{R}^V$ *and* $\alpha_i \geq 0$ $(i = 1, \cdots, k)$.

For any nonempty set $S \subseteq \mathbf{R}^V$, we define the *support function* $\delta_S^* : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of S by

$$\delta_S^*(p) = \sup_{x \in S} \langle p, x\rangle \qquad (p \in \mathbf{R}^V). \tag{23}$$

For any positively homogeneous function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, we define the set $S_f \subseteq \mathbf{R}^V$ by

$$S_f = \{x \in \mathbf{R}^V \mid \langle p, x\rangle \leq f(p)\ (\forall p \in \mathbf{R}^V)\}. \tag{24}$$

THEOREM 4.3 ([41, Section 13]).
(i) *For any nonempty polyhedron* S $(\subseteq \mathbf{R}^V)$, δ_S^* *is a positively homogeneous polyhedral convex function with* $\mathrm{dom}\,\delta_S^* \neq \emptyset$, *and* $S = S_{\delta_S^*}$.
(ii) *For any positively homogeneous polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, S_f *is a nonempty polyhedron, and* $f = \delta_{S_f}^*$.
(iii) *The mappings* $S \mapsto \delta_S^*$ *and* $f \mapsto S_f$ *provide a one-to-one correspondence between nonempty polyhedra* S $(\subseteq \mathbf{R}^V)$ *and positively homogeneous polyhedral convex functions* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, *and are the inverse of each other.*

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$. For any $x \in \mathrm{dom}\, f$ and $y \in \mathbf{R}^V$, we define the *(one-sided) directional derivative* of f at x with respect to y by

$$f'(x; y) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha}$$

if the limit exists. Note that if f is polyhedral convex, then $f'(x; y)$ is well-defined for any $x \in \mathrm{dom}\, f$ and $y \in \mathbf{R}^V$ by $f'(x; y) = \{f(x + \alpha y) - f(x)\}/\alpha$ with a sufficiently small $\alpha > 0$. The *subdifferential* $\partial f(x)$ of f at x is defined by

$$\partial f(x) = \{p \in \mathbf{R}^V \mid f(y) \geq f(x) + \langle p, y - x \rangle \ (\forall y \in \mathbf{R}^V)\},$$

and each vector in $\partial f(x)$ is called a *subgradient* of f at x.

THEOREM 4.4 ([41, Theorem 23.10]). *Let* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$, *and* $x \in \mathrm{dom}\, f$. *Then,* $\partial f(x)$ *is a nonempty polyhedral convex set, and* $f'(x; \cdot)$ *is a positively homogeneous polyhedral convex function such that* $\mathrm{dom}\, f'(x; \cdot) \neq \emptyset$, $f'(x; \cdot) > -\infty$, *and* $f'(x; \cdot) = \delta_{\partial f(x)}^*$.

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be any function. We denote by $\arg\min f$ the set of minimizers of f, i.e.,

$$\arg\min f = \{x \in \mathbf{R}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{R}^V)\}.$$

Note that $\arg\min f$ can be empty. For any $p \in \mathbf{R}^V$, the function $f[p] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f[p](x) = f(x) + \langle p, x \rangle \qquad (x \in \mathbf{R}^V). \tag{25}$$

THEOREM 4.5 ([41, Theorem 23.5]). *Let* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$*. For any* $x \in \mathbf{R}^V$ *and* $p \in \mathbf{R}^V$*, we have*

$$\begin{aligned} p \in \partial f(x) \iff & x \in \arg\min f[-p] \\ \iff & x \in \partial f^{\bullet}(p) \iff p \in \arg\min f^{\bullet}[-x]. \end{aligned}$$

For $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$, the *infimum convolution* (or simply the *convolution*) $f_1 \Box f_2 : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$(f_1 \Box f_2)(x) = \inf \left\{ f_1(x_1) + f_2(x_2) \;\middle|\; \begin{array}{c} x_1, x_2 \in \mathbf{R}^V, \\ x_1 + x_2 = x \end{array} \right\} \quad (x \in \mathbf{R}^V). \tag{26}$$

It is easy to verify that if $f_1 \Box f_2$ does not take the value $-\infty$, then

$$\mathrm{dom}\,(f_1 \Box f_2) = \mathrm{dom}\, f_1 + \mathrm{dom}\, f_2.$$

THEOREM 4.6 ([41, Sections 16, 19, and 20]). *Let* $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$ *be polyhedral convex functions with* $\mathrm{dom}\, f_i \neq \emptyset$*. Then,* $f_1 \Box f_2$ *is also polyhedral convex and*

$$(f_1 \Box f_2)^{\bullet} = f_1^{\bullet} + f_2^{\bullet}.$$

Moreover, if $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \neq \emptyset$ *then it holds that*

$$(f_1 + f_2)^{\bullet} = f_1^{\bullet} \Box f_2^{\bullet}.$$

We consider the special case of one-dimensional functions. A function $f : \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\}$ is called *piecewise-linear* if $\mathrm{dom}\, f$ is a closed set and can be divided into a finite number of subintervals on each of which f is linear. We denote by $\mathcal{C}^1$ the class of piecewise-linear convex functions (cf. (5)). Note that a piecewise-linear convex function is nothing but a one-dimensional polyhedral convex function. For $f : \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\}$ and $x \in \mathrm{dom}\, f$, we denote $f'_+(x) = f'(x; 1)$, $f'_-(x) = -f'(x; -1)$. If f is piecewise-linear, then the value $f'_+(x)$ and $f'_-(x)$ are well-defined for any $x \in \mathrm{dom}\, f$ and

$$\lim_{y \downarrow x} f'_+(y) = \lim_{y \downarrow x} f'_-(y) = f'_+(x), \quad \lim_{y \uparrow x} f'_+(y) = \lim_{y \uparrow x} f'_-(y) = f'_-(x).$$

4.2. Polyhedral M-convex Functions

A polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \quad (0 \leq \forall \alpha \leq \alpha_0).$$

We denote

$$\begin{aligned} \mathcal{M} &= \{f \mid f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{polyhedral M-convex}\}, \\ {}_0\mathcal{M} &= \{f \mid f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{positively homogeneous} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{polyhedral M-convex}\}. \end{aligned}$$

It may be obvious from the definition that polyhedral M-convex functions are a quantitative extension of M-convex polyhedra.

THEOREM 4.7.
(i) *For a function* $f : \mathbf{R}^V \to \{0, +\infty\}$*, we have* $f \in \mathcal{M} \iff \mathrm{dom}\, f \in \mathcal{M}_0$.
(ii) *For* $f \in \mathcal{M}$*, we have* $\mathrm{dom}\, f \in \mathcal{M}_0$.

4.2.1. Axioms for Polyhedral M-convex Functions

We consider two slightly different exchange axioms, where the former is weaker and the latter is stronger than (M-EXC).

(M-EXC$_\mathbf{w}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)).$$

(M-EXC$_\mathbf{s}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v))$$

for any $\alpha \in \mathbf{R}$ with $0 \leq \alpha \leq \{x(u) - y(u)\}/2k$, where $k = |\mathrm{supp}^-(x-y)|$.

THEOREM 4.8. *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, (M-EXC) $\iff$ (M-EXC$_\mathrm{w}$).

Proof. We show (M-EXC$_{\rm w}$) $\Rightarrow$ (M-EXC) only. Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x-y)$. Then, there exist $v \in \mathrm{supp}^-(x-y)$, $\alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \tag{27}$$

for $\alpha = \alpha_0$. The convexity of f implies (27) for any $\alpha \in [0, \alpha_0]$. ▮

LEMMA 4.9. *Let $f \in \mathcal{M}$, $x, y \in \mathrm{dom}\, f$ be distinct vectors, $u \in \mathrm{supp}^+(x-y)$, and $\mathrm{supp}^-(x-y) = \{v_1, v_2, \cdots, v_k\}$ $(k = |\mathrm{supp}^-(x-y)|)$. Then, there exist sequences $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$ $(\subseteq \mathrm{dom}\, f)$ and $\{\alpha_i\}_{i=1}^k$ $(\subseteq \mathbf{R}_+)$ satisfying $\sum_{i=1}^k \alpha_i = \{x(u) - y(u)\}/2$ and the following conditions:*

$$x_0 = x, \ y_0 = y,$$
$$x_i = x_{i-1} - \alpha_i(\chi_u - \chi_{v_i}), \ y_i = y_{i-1} + \alpha_i(\chi_u - \chi_{v_i}) \quad (i = 1, \cdots, k), \tag{28}$$
$$f(x_{i-1}) + f(y_{i-1}) \geq f(x_i) + f(y_i) \quad (i = 1, \cdots, k). \tag{29}$$

Proof. Put $x_0 = x$, $y_0 = y$. For $i = 1, 2, \cdots, k$, we define $\alpha_i \in \mathbf{R}$ and $x_i, y_i \in \mathbf{R}^V$ iteratively by the following equations:

$$\begin{aligned} \alpha_i = \sup\{\alpha \in \mathbf{R}_+ \mid \alpha \leq \min\{x_{i-1}(u) - y_{i-1}(u), y_{i-1}(v_i) - x_{i-1}(v_i)\}/2, \\ f(x_{i-1} - \alpha(\chi_u - \chi_{v_i})) + f(y_{i-1} + \alpha(\chi_u - \chi_{v_i})) \\ \leq f(x_{i-1}) + f(y_{i-1})\}, \end{aligned}$$
$$x_i = x_{i-1} - \alpha_i(\chi_u - \chi_{v_i}), \quad y_i = y_{i-1} + \alpha_i(\chi_u - \chi_{v_i}).$$

Then, the sequences $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$, and $\{\alpha_i\}_{i=1}^k$ satisfy the conditions (28) and (29). In the following, we show $\sum_{i=1}^k \alpha_i = \{x(u) - y(u)\}/2$.

Assume, to the contrary, that $\sum_{i=1}^k \alpha_i < \{x(u) - y(u)\}/2$. Since $u \in \mathrm{supp}^+(x_k - y_k)$, there exist some $v_i \in \mathrm{supp}^-(x_k - y_k) \subseteq \mathrm{supp}^-(x-y)$ and $\alpha_0 > 0$ such that

$$f(x_k) + f(y_k) \geq f(x_k - \alpha(\chi_u - \chi_{v_i})) + f(y_k + \alpha(\chi_u - \chi_{v_i})) \tag{30}$$

for any $\alpha \in [0, \alpha_0]$. Here, $v_i \neq v_k$ holds by the choice of x_k and y_k. By Theorem 4.14 to be shown later, we have

$$\begin{aligned} &\{f(x_i - \alpha(\chi_u - \chi_{v_i})) - f(x_i)\} + \{f(y_i + \alpha(\chi_u - \chi_{v_i})) - f(y_i)\} \\ &\leq \{f(x_k - \alpha(\chi_u - \chi_{v_i})) - f(x_k)\} + \{f(y_k + \alpha(\chi_u - \chi_{v_i})) - f(y_k)\} \leq 0 \end{aligned}$$

for any $\alpha \in [0, \alpha_0]$, where the second inequality is by (30). This contradicts the definitions of x_i and y_i. ▮

THEOREM 4.10. *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, (M-EXC) $\iff$ (M-EXC$_\mathrm{s}$).

Proof. It suffices to prove (M-EXC) $\Rightarrow$ (M-EXC$_\mathrm{s}$). Assume (M-EXC) for f. Let $x, y \in \mathrm{dom}\, f$ be distinct vectors, and $u \in \mathrm{supp}^+(x - y)$. By Lemma 4.9, there exist sequences $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$ ($\subseteq \mathrm{dom}\, f$) and $\{\alpha_i\}_{i=1}^k$ ($\subseteq \mathbf{R}_+$) satisfying $\sum_{i=1}^k \alpha_i = \{x(u) - y(u)\}/2$, (28), and (29). Let i_* be an integer with $1 \leq i_* \leq k$ such that

$$\alpha_{i_*} = \max_{1 \leq i \leq k} \alpha_i \ (\geq \{x(u) - y(u)\}/2k).$$

From Theorem 4.14 to be shown later, we obtain the inequalities:

$$\begin{aligned} &\{f(x - \alpha_{i_*}(\chi_u - \chi_{v_{i_*}})) - f(x)\} + \{f(y + \alpha_{i_*}(\chi_u - \chi_{v_{i_*}})) - f(y)\} \\ &\quad \leq \ \{f(x_{i_*}) - f(x_{i_*-1})\} + \{f(y_{i_*}) - f(y_{i_*-1})\} \ \leq \ 0, \end{aligned}$$

where the second inequality is by (29). From the convexity of f follows

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_{v_{i_*}})) + f(y + \alpha(\chi_u - \chi_{v_{i_*}})) \quad (\forall \alpha \in [0, \alpha_{i_*}]).$$

∎

We can rewrite (M-EXC) in terms of directional derivatives. For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ and $x \in \mathrm{dom}\, f$, define $f'(x; \cdot, \cdot) : V \times V \to \mathbf{R} \cup \{+\infty\}$ by

$$f'(x; v, u) = f'(x; \chi_v - \chi_u) \qquad (u, v \in V).$$

Note that $f(x - \alpha(\chi_u - \chi_v)) - f(x) = \alpha f'(x; v, u)$ for sufficiently small $\alpha > 0$. We consider (M-EXC$'$):

(M-EXC$'$) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$f'(x, v, u) + f'(y, u, v) \leq 0.$$

THEOREM 4.11. *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, (M-EXC) $\iff$ (M-EXC$'$).

4.2.2. Fundamental Properties of Polyhedral M-convex Functions

In this section, we show various properties of polyhedral M-convex functions.

Global optimality of a polyhedral M-convex function is characterized by local optimality.

THEOREM 4.12. *Let* $f \in \mathcal{M}$ *and* $x \in \mathrm{dom}\, f$. *Then,* $f(x) \leq f(y)$ $(\forall y \in \mathbf{R}^V) \iff f'(x; v, u) \geq 0$ $(\forall u, v \in V)$.

Proof. We show the "$\Longleftarrow$" direction only. Assume, to the contrary, that $f(x_0) < f(x)$ holds for some $x_0 \in \mathbf{R}^V$. Put $S = \{y \in \mathbf{R}^V \mid f(y) \leq f(x_0)\}$. Let $x_* \in S$ be a vector such that $\|x_* - x\|_1 = \inf_{y \in S} \|y - x\|_1$. By (M-EXC) applied to x, x_* and some $u \in \mathrm{supp}^+(x - x_*)$, there exists $v \in \mathrm{supp}^-(x - x_*)$ and a sufficiently small $\alpha > 0$ such that

$$\begin{aligned} f(x_*) - f(x_* + \alpha(\chi_u - \chi_v)) &\geq f(x - \alpha(\chi_u - \chi_v)) - f(x) \\ &\geq \alpha f'(x; v, u) \geq 0. \end{aligned}$$

Hence, we have $f(x_* + \alpha(\chi_u - \chi_v)) \leq f(x_*) \leq f(x_0)$, which contradicts the choice of x_* since $\|(x_* + \alpha(\chi_u - \chi_v)) - x\|_1 = \|x_* - x\|_1 - 2\alpha$. ▌

A polyhedral M-convex function has supermodularity when projected onto the hyperplane $\{x \in \mathbf{R}^V \mid x(w_0) = 0\}$ $(w_0 \in V)$, as stated in Theorem 4.14.

LEMMA 4.13. *Let* $f \in \mathcal{M}$, $w_0 \in V$, *and* $x \in \mathbf{R}^V$. *Then, for any* $u, v \in V$ *and* $\lambda, \mu \geq 0$ *we have*

$$\begin{aligned} &f(x + \lambda(\chi_u - \chi_{w_0})) + f(x + \mu(\chi_v - \chi_{w_0})) \\ &\quad \leq f(x) + f(x + \lambda(\chi_u - \chi_{w_0}) + \mu(\chi_v - \chi_{w_0})). \end{aligned} \tag{31}$$

Proof. Fix $u, v \in V$ and $\lambda, \mu \geq 0$. For λ', $\mu' \in \mathbf{R}_+$, we denote $x(\lambda', \mu') = x + \lambda'(\chi_u - \chi_{w_0}) + \mu'(\chi_v - \chi_{w_0})$. We may assume that $x(0,0), x(\lambda, \mu) \in \mathrm{dom}\, f$. Then, it follows from Theorem 3.9 that

$$x(\lambda', 0),\ x(\lambda', \mu) \in \mathrm{dom}\, f \quad (0 \leq \forall \lambda' \leq \lambda).$$

Define the functions $f_1, f_2 : [0, \lambda] \to \mathbf{R}$ by

$$f_1(\lambda') = f(x(\lambda', 0)), \quad f_2(\lambda') = f(x(\lambda', \mu)) \qquad (0 \leq \lambda' \leq \lambda).$$

Let γ be any value with $0 < \gamma \leq \lambda$. For any $\beta \in [0, \gamma)$, (M-EXC$'$) yields $(f_1)'_+(\beta) - (f_2)'_-(\gamma) \leq 0$ since $\mathrm{supp}^+(x(\gamma, \mu) - x(\beta, 0)) = \{u, v\}$ and

$\mathrm{supp}^-(x(\gamma,\mu) - x(\beta,0)) = \{w_0\}$. This implies

$$(f_2 - f_1)'_-(\gamma) = (f_2)'_-(\gamma) - (f_1)'_-(\gamma) = (f_2)'_-(\gamma) - \lim_{\beta\uparrow\gamma}(f_1)'_+(\beta) \geq 0$$

for any γ with $0 < \gamma \leq \lambda$. Therefore, the piecewise-linear function $f_2 - f_1$ is nondecreasing on $[0, \lambda]$, from which we have

$$\begin{aligned} f(x(\lambda,\mu)) - f(x(\lambda,0)) &= f_2(\lambda) - f_1(\lambda) \\ &\geq f_2(0) - f_1(0) = f(x(0,\mu)) - f(x(0,0)), \end{aligned}$$

i.e., the inequality (31) holds. ▮

THEOREM 4.14. *Let $f \in \mathcal{M}$, $w_0 \in V$, and $x \in \mathbf{R}^V$. For any $X, Y \subseteq V \setminus \{w_0\}$ with $X \cap Y = \emptyset$ and $\lambda_w \geq 0$ $(w \in X \cup Y)$, we have*

$$\begin{aligned} & f(x + \sum_{w\in X} \lambda_w(\chi_w - \chi_{w_0})) + f(x + \sum_{w\in Y} \lambda_w(\chi_w - \chi_{w_0})) \\ \leq\ & f(x) + f(x + \sum_{w\in X\cup Y} \lambda_w(\chi_w - \chi_{w_0})). \end{aligned} \tag{32}$$

Proof. We show the inequality (32) by induction on the cardinality of the set $X \cup Y$. It suffices to consider the case when $X \neq \emptyset$ and $Y \neq \emptyset$. Note that the case when $|X| = |Y| = 1$ is already shown in Lemma 4.13. Therefore, we may assume that $|Y| \geq 2$. Let $v \in Y$. We assume that $x \in \mathrm{dom}\, f$ and $x + \sum_{w\in X\cup Y} \lambda_w(\chi_w - \chi_{w_0}) \in \mathrm{dom}\, f$. Then, we have $x' = x + \lambda_v(\chi_v - \chi_{w_0}) \in \mathrm{dom}\, f$ by Theorem 3.9. Hence, the inductive assumption yields

$$\begin{aligned} & f(x + \sum_{w\in X} \lambda_w(\chi_w - \chi_{w_0})) - f(x) \\ \leq\ & f(x + \sum_{w\in X\cup\{v\}} \lambda_w(\chi_w - \chi_{w_0})) - f(x + \lambda_v(\chi_v - \chi_{w_0})) \\ \leq\ & f(x + \sum_{w\in X\cup Y} \lambda_w(\chi_w - \chi_{w_0})) - f(x + \sum_{w\in Y} \lambda_w(\chi_w - \chi_{w_0})). \end{aligned}$$

▮

Directional derivative functions and subdifferentials of a polyhedral M-convex function have nice structures such as M/L-convexity, and they can be explicitly described by certain distance functions with triangle inequality (cf. Theorem 4.15 (i)).

For any distance function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ (i.e., $\gamma(v,v) = 0$ for all $v \in V$), we define $f_\gamma : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_\gamma(x) = \inf \left\{ \sum_{u,v \in V} \lambda_{uv}\gamma(u,v) \;\middle|\; \begin{array}{l} \sum_{u,v\in V} \lambda_{uv}(\chi_v - \chi_u) = x, \\ \lambda_{uv} \geq 0 \ (u,v \in V) \end{array} \right\} \quad (x \in \mathbf{R}^V). \tag{33}$$

THEOREM 4.15. *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$.*
(i) *The function $\gamma_x : V \times V \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$\gamma_x(u,v) = f'(x; v, u) \qquad (u, v \in V) \tag{34}$$

satisfies $\gamma_x(v,v) = 0$ ($\forall v \in V$) and the triangle inequality (19), i.e., $\gamma_x \in \mathcal{T}$.
(ii) *We have $f'(x;\cdot) = f_{\gamma_x}$ and $f'(x;\cdot) \in {}_0\mathcal{M}$.*

Proof. (i): For $u, v, w \in V$ and a sufficiently small $\alpha > 0$, we have

$$\begin{aligned} &\alpha\gamma_x(u,v) + \alpha\gamma_x(v,w) \\ &\quad = \{f(x + \alpha(\chi_v - \chi_u)) - f(x)\} + \{f(x + \alpha(\chi_w - \chi_v)) - f(x)\} \\ &\quad \geq f(x + \alpha(\chi_w - \chi_u)) - f(x) \ = \ \alpha\gamma_x(u,w), \end{aligned}$$

where the inequality is by Lemma 4.13. Obviously, γ_x is a distance function.

(ii) The proof is postponed after Theorem 4.19. ▮

L-convexity appears in subdifferentials of a polyhedral M-convex function.

THEOREM 4.16. *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$.*
(i) *$\partial f(x) \in \mathcal{L}_0$ and $\partial f(x)$ is represented as*

$$\partial f(x) = \mathrm{D}(\gamma_x) = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq f'(x; v, u) \ (u, v \in V)\}, \tag{35}$$

where $\gamma_x : V \times V \to \mathbf{R} \cup \{+\infty\}$ is given by (34).
(ii) *For any $y \in \mathbf{R}^V$ we have $f(y) - f(x) \geq \sup_{p \in \partial f(x)} \langle p, y - x\rangle = f_{\gamma_x}(y - x)$.*

Proof. The equation (35) follows from Theorems 4.5 and 4.12. The L-convexity of $\partial f(x)$ is immediate from (35) and Theorem 4.15 (i). The claim (ii) is immediate from (i) and the linear programming duality. ▮

The next theorem shows that each face of the epigraph of a polyhedral M-convex function is an M-convex polyhedron when it is projected to $\mathbf{R}^V$. The proof is obvious and therefore omitted.

THEOREM 4.17. *For $f \in \mathcal{M}$ and $p \in \mathbf{R}^V$, we have* $\arg\min f[-p] \in \mathcal{M}_0$ *if* $\inf f[-p] > -\infty$.

The class of polyhedral M-convex functions is closed under various fundamental operations. Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. For any subset $U \subseteq V$, we define $f_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ by

$$f_U(y) = f(y, \mathbf{0}_{V \setminus U}) \qquad (y \in \mathbf{R}^U). \tag{36}$$

THEOREM 4.18. *Let $f, f_1, f_2 \in \mathcal{M}$.*
(1) For $p \in \mathbf{R}^V$, the function $f[-p] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ given by (25) is polyhedral M-convex.
(2) For $a \in \mathbf{R}^V$ and $\nu > 0$, the functions $\nu \cdot f(a - x)$ and $\nu \cdot f(a + x)$ are polyhedral M-convex in x.
(3) For any $U \subseteq V$, the function $f_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ is polyhedral M-convex if $\mathrm{dom}\, f_U \neq \emptyset$.
(4) For $\varphi_v \in \mathcal{C}^1$ $(v \in V)$, the function $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x) = f(x) + \sum_{v \in V} \varphi_v(x(v)) \quad (x \in \mathbf{R}^V)$$

is polyhedral M-convex if $\mathrm{dom}\, \tilde{f} \neq \emptyset$.
(5) For any $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a \leq b$, the restriction $f_{[a,b]} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of f defined by

$$f_{[a,b]}(x) = \begin{cases} f(x) & (x \in [a,b]), \\ +\infty & (x \notin [a,b]) \end{cases} \tag{37}$$

is polyhedral M-convex if $\mathrm{dom}\, f \cap [a,b] \neq \emptyset$.
(6) The convolution $f_1 \Box f_2 : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ defined by (26) is polyhedral M-convex, provided $(f_1 \Box f_2)(x_0)$ is finite for some $x_0 \in \mathbf{R}^V$.
(7) For $U \subseteq V$, the function $\hat{f} : \mathbf{R} \times \mathbf{R}^U \to \mathbf{R} \cup \{\pm\infty\}$ defined by

$$\hat{f}(y_0, y) = \inf\{f(x) \mid x(v) = y(v) \ (v \in U),\ y_0 = x(V \setminus U)\} \tag{38}$$

is polyhedral M-convex if $\hat{f}(y_0', y')$ is finite (i.e., $-\infty < \hat{f}(y_0', y') < +\infty$) for some $(y_0', y') \in \mathbf{R} \times \mathbf{R}^U$.

Proof. (1) to (5) are easy to prove. The proofs of (6) and (7) are given in Sections 5 and 7.3, respectively. ∎

A more general and stronger transformation, called "network induction," is explained in Section 7.3.

4.2.3. Positively Homogeneous Polyhedral M-convex Functions

This section clarifies the relationship of positively homogeneous polyhedral M-convex functions to distance functions with triangle inequalities, and also to L-convex polyhedra.

For a positively homogeneous polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathbf{0} \in \mathrm{dom}\, f$, define $\gamma_f : V \times V \to \mathbf{R} \cup \{+\infty\}$ by

$$\gamma_f(u,v) = f'(\mathbf{0}; v, u) \ (= f(\chi_v - \chi_u)) \qquad (u, v \in V). \tag{39}$$

Recall the definition of f_γ in (33).

THEOREM 4.19.
(i) *For $f \in {}_0\mathcal{M}$, we have $\gamma_f \in \mathcal{T}$ and $f_{\gamma_f} = f$.*
(ii) *For $\gamma \in \mathcal{T}$, we have $f_\gamma \in {}_0\mathcal{M}$ and $\gamma_{f_\gamma} = \gamma$.*
(iii) *The mappings $f \mapsto \gamma_f$ $(f \in {}_0\mathcal{M})$ and $\gamma \mapsto f_\gamma$ $(\gamma \in \mathcal{T})$ provide a one-to-one correspondence between ${}_0\mathcal{M}$ and $\mathcal{T}$, and are the inverse of each other.*

Proof. (i): By Theorem 4.15 (i) and Theorem 4.16 (ii), we have $\gamma_f \in \mathcal{T}$ and $f \geq f_{\gamma_f}$. For any $x \in \mathbf{R}^V$ and $\{\lambda_{uv}\}_{u,v \in V}$ satisfying $\sum_{u,v\in V} \lambda_{uv}(\chi_v - \chi_u) = x$ and $\lambda_{uv} \geq 0$ $(u, v \in V)$, Theorem 4.2 implies that

$$\begin{aligned}\sum_{u,v\in V} \lambda_{uv}\gamma_f(u,v) &= \sum_{u,v\in V} \lambda_{uv} f(\chi_v - \chi_u) \\ &\geq f(\sum_{u,v\in V} \lambda_{uv}(\chi_v - \chi_u)) \ = \ f(x).\end{aligned}$$

Hence, we have $f_{\gamma_f} = f$.

(ii): Polyhedral M-convexity of f_γ can be shown as a special case of Example 2.4. To be more precise, we have $f_\gamma(x) = f(x)$ $(\forall x \in \mathbf{R}^V)$ for the function f in Example 2.4, where $T = V$, $A = \{(u,v) \mid u, v \in V,\ u \neq v\}$, and

$$f_{(u,v)}(\xi) = \begin{cases} \gamma(u,v)\xi & (\xi \geq 0), \\ +\infty & (\xi < 0), \end{cases} \qquad ((u,v) \in A).$$

For any $u, v \in V$, we have $\gamma_{f_\gamma}(u,v) = f_\gamma(\chi_v - \chi_u)$, which is equal to the shortest path distance from u to v in the directed graph $G = (V, A)$, where $A = V \times V$ and the length of an arc $(u,v) \in A$ is given by $\gamma(u,v)$. Since $\gamma \in \mathcal{T}$, it holds $\gamma_{f_\gamma}(u,v) = \gamma(u,v)$ $(\forall u, v \in V)$.

(iii): Clear from (i) and (ii). ▮

We now prove the polyhedral M-convexity of the directional derivative function $f'(x;\cdot)$.

Proof of Theorem 4.15 (ii) By Theorems 4.15 (i) and 4.19, it suffices to show $f'(x;\cdot) = f_{\gamma_x}$, which can be done in the same way as the proof of Theorem 4.19 (i). ▮

As immediate consequences of Theorems 3.23, 4.3, and 4.19, we obtain a one-to-one correspondence between positively homogeneous polyhedral M-convex functions and L-convex polyhedra. Recall the notation S_f and δ_S^* in (23) and (24).

THEOREM 4.20.
(i) *For $f \in {}_0\mathcal{M}$, we have $S_f \in \mathcal{L}_0$ and $\delta_{S_f}^* = f$.*
(ii) *For $D \in \mathcal{L}_0$, we have $\delta_D^* \in {}_0\mathcal{M}$ and $S_{\delta_D^*} = D$.*
(iii) *The mappings $f \mapsto S_f$ $(f \in {}_0\mathcal{M})$ and $D \mapsto \delta_D^*$ $(D \in \mathcal{L}_0)$ provide a one-to-one correspondence between ${}_0\mathcal{M}$ and $\mathcal{L}_0$, are the inverse of each other.*

4.2.4. Polyhedral M♮-convex Functions

We introduce a variant of polyhedral M-convex functions, called polyhedral M♮-convex functions. From Theorem 3.2 and 4.7 (ii), we see that the effective domain of a polyhedral M-convex function is contained in a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = r\}$ for some $r \in \mathbf{R}$. Therefore, no information is lost when a polyhedral M-convex function is projected onto a $(|V|-1)$-dimensional space. A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called *polyhedral M♮-convex* if the function $\tilde{f} : \mathbf{R}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(V)), \\ +\infty & (x_0 \neq -x(V)), \end{cases} \qquad ((x_0, x) \in \mathbf{R}^{\tilde{V}})$$

is a polyhedral M-convex function, where $\tilde{V} = \{v_0\} \cup V$. Polyhedral M♮-convex functions are essentially equivalent to polyhedral M-convex functions, whereas the class of polyhedral M♮-convex functions properly contains that of polyhedral M-convex functions. M♮-convexity was originally introduced in [37] as a concept for functions defined over the integer lattice (see also [18]).

Polyhedral M♮-convex functions can be characterized by the following exchange property (cf. [37]):

(M$^\natural$-EXC) $\forall x, y \in \operatorname{dom} f$, $\forall u \in \operatorname{supp}^+(x-y)$, either (i) or (ii) (or both) holds:
(i) $\exists v \in \operatorname{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \quad (0 \leq \forall \alpha \leq \alpha_0),$$

(ii) $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha\chi_u) + f(y + \alpha\chi_u) \quad (0 \leq \forall \alpha \leq \alpha_0).$$

THEOREM 4.21. *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\operatorname{dom} f \neq \emptyset$. *Then, f is M$^\natural$-convex if and only if it satisfies* (M$^\natural$-EXC).

Proof. Proof is given in Section 5. ▮

EXAMPLE 4.22 (separable-convex functions). Let $\mathcal{A} \subseteq 2^V$ be a laminar family, i.e., $\mathcal{A}$ is a family of subsets of V such that for any $X, Y \in \mathcal{A}$ at least one of $X \setminus Y$, $Y \setminus X$ and $X \cap Y$ is empty. For $f_X \in \mathcal{C}^1$ $(X \in \mathcal{A})$, the function $f_{\mathcal{A}} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ given by

$$f_{\mathcal{A}}(x) = \sum_{X \in \mathcal{A}} f_X(x(X)) \qquad (x \in \mathbf{R}^V)$$

is polyhedral M$^\natural$-convex if $\operatorname{dom} f_{\mathcal{A}} \neq \emptyset$ [6]. This can be proved similarly to Example 2.4.

Let $B \subseteq \mathbf{R}^V$ be an M$^\natural$-convex polyhedron. For $f_0 \in \mathcal{C}^1$ and $f_v \in \mathcal{C}^1$ $(v \in V)$, the function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} f_0(x(V)) + \sum_{v \in V} f_v(x(v)) & (x \in B), \\ +\infty & (x \notin B) \end{cases}$$

is polyhedral M$^\natural$-convex if $\operatorname{dom} f \neq \emptyset$. ▮

Another class of polyhedral M$^\natural$-convex functions arises from the minimum cost flow problem, as in Example 2.4.

From the definition of polyhedral M$^\natural$-convex functions, every property of polyhedral M-convex functions can be restated in terms of polyhedral M$^\natural$-convex functions, and vice versa. We present some properties below, which seem more natural when restated in terms of polyhedral M$^\natural$-convex functions.

For any function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and any subset $U \subseteq V$, we define the *projection* $f^U : \mathbf{R}^U \to \mathbf{R} \cup \{\pm\infty\}$ of f to U as

$$f^U(y) = \inf_{z \in \mathbf{R}^{V \setminus U}} f(y, z) \qquad (y \in \mathbf{R}^U). \tag{40}$$

THEOREM 4.23. *For any polyhedral M♮-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $U \subseteq V$, f^U is polyhedral M♮-convex if $f^U(x_0)$ is finite for some $x_0 \in \mathbf{R}^U$.*

Proof. Obvious from Theorem 4.18 (7). ▮

THEOREM 4.24. *If $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is polyhedral M♮-convex, then it satisfies the supermodular inequality:*

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \qquad (\forall x, y \in \mathbf{R}^V).$$

Proof. Immediate from Theorem 4.14. ▮

4.3. Polyhedral L-convex Functions

A polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \quad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$.

We denote

$$\begin{aligned} \mathcal{L} &= \{g \mid g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{polyhedral L-convex}\}, \\ {}_0\mathcal{L} &= \{g \mid g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{positively homogeneous} \\ &\qquad\qquad\qquad\qquad\qquad \text{polyhedral L-convex}\}. \end{aligned}$$

As is obvious from the definition, polyhedral L-convex functions are a quantitative generalization of L-convex polyhedra.

THEOREM 4.25.
(i) *For a function $g : \mathbf{R}^V \to \{0, +\infty\}$, $g \in \mathcal{L} \iff \mathrm{dom}\, g \in \mathcal{L}_0$.*
(ii) *For $g \in \mathcal{L}$, we have $\mathrm{dom}\, g \in \mathcal{L}_0$.*

4.3.1. Axioms for Polyhedral L-convex Functions

We first show that (LF1) and (LF2) are equivalent to local properties ($\mathrm{LF1_{loc}}$) and ($\mathrm{LF2_{loc}}$):

($\mathbf{LF1_{loc}}$) $\forall p \in \mathrm{dom}\, g$, $\exists \varepsilon > 0$ such that

$$g(q_1) + g(q_2) \geq g(q_1 \wedge q_2) + g(q_1 \vee q_2) \quad (\forall q_1, q_2 \in \overline{\mathrm{N}_\infty(p, \varepsilon)}).$$

($\mathbf{LF2_{loc}}$) $\forall p \in \mathrm{dom}\, g$, $\exists \varepsilon > 0$, $\exists r \in \mathbf{R}$ such that

$$q \in \overline{\mathrm{N}_\infty(p, \varepsilon)},\ q + \lambda \mathbf{1} \in \overline{\mathrm{N}_\infty(p, \varepsilon)}\ (\lambda \in \mathbf{R}) \Longrightarrow g(q + \lambda \mathbf{1}) = g(q) + \lambda r.$$

THEOREM 4.26. *Let* $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, g \neq \emptyset$. *Then,*
(i) (LF1) $\Longleftrightarrow$ ($\mathrm{LF1_{loc}}$). (ii) (LF2) $\Longleftrightarrow$ ($\mathrm{LF2_{loc}}$).

Proof. (i): We show ($\mathrm{LF1_{loc}}$) $\Rightarrow$ (LF1) only. Assume ($\mathrm{LF1_{loc}}$) for g. Then, Theorem 3.20 (i) implies (LS1) for $\mathrm{dom}\, g$. For $p \in \mathrm{dom}\, g$, we denote by $\varepsilon(p)$ the value ε in ($\mathrm{LF1_{loc}}$) associated with p.

Let $p, p' \in \mathrm{dom}\, g$, and recall the definition of $\mathrm{Box}[p, p']$ in Section 3.1. Since $\mathrm{Box}[p, p'] \cap \mathrm{dom}\, g$ is a compact set, there exist $q_1, q_2, \cdots, q_m \in \mathrm{Box}[p, p'] \cap \mathrm{dom}\, g$ ($m \geq 1$) such that $\mathrm{Box}[p, p'] \cap \mathrm{dom}\, g \subseteq \bigcup_{i=1}^m \mathrm{N}_\infty(q_i, \varepsilon(q_i))$. We show the submodular inequality

$$g(p) + g(p') \geq g(p \wedge p') + g(p \vee p') \tag{41}$$

by induction on the integer m. We may assume $m \geq 2$, since otherwise (41) holds immediately.

For $\lambda, \mu \in [0, 1]$, put $p(\lambda, \mu) = p + \lambda\{(p \vee p') - p\} + \mu\{(p \wedge p') - p\}$. Then, for $w \in V$ we have

$$p(\lambda, \mu)(w) = \begin{cases} (1-\lambda)p(w) + \lambda p'(w) & (w \in \mathrm{supp}^+(p' - p)), \\ p(w) & (w \in V \setminus \mathrm{supp}(p' - p)), \\ (1-\mu)p(w) + \mu p'(w) & (w \in \mathrm{supp}^-(p' - p)). \end{cases} \tag{42}$$

Furthermore, for any $\lambda, \mu \in [0, 1]$ we have

$$p(\lambda, \mu) \in \mathrm{conv}\{p,\ p',\ p \wedge p',\ p \vee p'\} \subseteq \mathrm{Box}[p, p'] \cap \mathrm{dom}\, g.$$

We may assume that $p \in \mathrm{N}_\infty(q_1, \varepsilon(q_1))$. Put $p_* = p(\lambda_*, \mu_*)$, where

$$\begin{aligned} \lambda_* &= \sup\{\lambda \in [0, 1] \mid p(\lambda, 0) \in \mathrm{N}_\infty(q_1, \varepsilon(q_1))\}, \\ \mu_* &= \sup\{\mu \in [0, 1] \mid p(0, \mu) \in \mathrm{N}_\infty(q_1, \varepsilon(q_1))\}. \end{aligned}$$

Then, we have $p_* \in \overline{\mathrm{N}_\infty(q_1, \varepsilon(q_1))}$, implying

$$g(p) + g(p_*) \geq g(p \wedge p_*) + g(p \vee p_*). \tag{43}$$

In the following, we assume $\lambda_* < 1$ and $\mu_* < 1$, since the other cases can be shown similarly and more easily. Then, we have

$$\begin{aligned}
\mathrm{Box}[p \vee p_*, p' \vee p_*] \cap \mathrm{dom}\, g &\subseteq \bigcup_{i=2}^{m} \mathrm{N}_\infty(q_i, \varepsilon(q_i)), \\
\mathrm{Box}[p \wedge p_*, p' \wedge p_*] \cap \mathrm{dom}\, g &\subseteq \bigcup_{i=2}^{m} \mathrm{N}_\infty(q_i, \varepsilon(q_i)), \\
\mathrm{Box}[p_*, p'] \cap \mathrm{dom}\, g &\subseteq \bigcup_{i=2}^{m} \mathrm{N}_\infty(q_i, \varepsilon(q_i)).
\end{aligned}$$

Hence, the inductive hypothesis implies that

$$\begin{aligned}
&g(p \vee p_*) + g(p' \vee p_*) \geq g(p_*) + g(p \vee p'), \\
&g(p \wedge p_*) + g(p' \wedge p_*) \geq g(p \wedge p') + g(p_*), \\
&g(p_*) + g(p') \geq g(p_* \wedge p') + g(p_* \vee p'),
\end{aligned}$$

where we note that

$$\begin{aligned}
&(p \vee p_*) \wedge (p' \vee p_*) = p_*, \qquad &(p \vee p_*) \vee (p' \vee p_*) = p \vee p', \\
&(p \wedge p_*) \wedge (p' \wedge p_*) = p \wedge p', \qquad &(p \wedge p_*) \vee (p' \wedge p_*) = p_*.
\end{aligned}$$

Combining the above inequalities with (43), we obtain the submodular inequality (41).

(ii) We show (LF2$_{\mathrm{loc}}$) $\Rightarrow$ (LF2) only. Assume (LF2$_{\mathrm{loc}}$) for g. Then, $\mathrm{dom}\, g$ satisfies (LS2) by Theorem 3.20 (ii). Let $x_i \in \mathbf{R}^V$ and $\lambda_i \in \mathbf{R}$ $(i = 1, \cdots, k)$ be such that $g(p) = \max_{1 \leq i \leq k}\{\langle p, x_i \rangle + \lambda_i\}$ $(p \in \mathrm{dom}\, g)$. As is shown later, the value $\langle \mathbf{1}, x_i \rangle$ is the same for all i. Thus, g satisfies (LF2).

We now prove that $\langle \mathbf{1}, x_i \rangle = \langle \mathbf{1}, x_j \rangle$ for any i, j. Let p_0 be any vector in $\mathrm{dom}\, g$, and define $\psi : \mathbf{R} \to \mathbf{R}$ by $\psi(\mu) = g(p_0 + \mu \mathbf{1})$ $(\mu \in \mathbf{R})$. Then, ψ is piecewise-linear convex. Moreover, we have $\psi'_+(\mu_+) = \max_{1 \leq i \leq k}\langle \mathbf{1}, x_i \rangle$ for a sufficiently large value of μ_+, and $\psi'_+(\mu_-) = \min_{1 \leq i \leq k}\langle \mathbf{1}, x_i \rangle$ for a sufficiently small value of μ_-. By (LF2$_{\mathrm{loc}}$), we see that ψ is a linear function. Hence, we have $\max_{1 \leq i \leq k}\langle \mathbf{1}, x_i \rangle = \psi'_+(\mu_+) = \psi'_+(\mu_-) = \min_{1 \leq i \leq k}\langle \mathbf{1}, x_i \rangle$. ∎

THEOREM 4.27. *Let $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ be such that $a \leq b$. Also, let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with* $\mathrm{dom}\, g = [a, b]$.
(i) *g is submodular $\iff$ for any $p \in \mathbf{R}^V$, any $u, v \in V$ $(u \neq v)$, we have*

$$g(p + \lambda\chi_u) + g(p + \mu\chi_v) \geq g(p) + g(p + \lambda\chi_u + \mu\chi_v) \quad (\forall \lambda, \mu \geq 0). \tag{44}$$

(ii) *The condition (44) can be replaced by the following:*

$$g(p + \lambda\chi_u) + g(p + \mu\chi_v) \geq g(p) + g(p + \lambda\chi_u + \mu\chi_v) \tag{45}$$

for any $\lambda \in [0, p_{l-1} - p_l]$ *and* $\mu \in [0, p_{m-1} - p_m]$, *where* $p_1 > p_2 > \cdots > p_k$ *are distinct values in* $\{p(v)\}_{v\in V}$, *and* $p(u) = p_l$, $p(v) = p_m$.

Proof. We show the "if" part of (i) only, and the other proofs are omitted. Let $p, q \in [a, b]$. Note that $p \wedge q, p \vee q \in [a, b]$. We show the submodular inequality for p, q by induction on the cardinality of $\mathrm{supp}(p-q)$. Without loss of generality, we assume $|\mathrm{supp}^-(p-q)| \geq 2$. Let $v \in \mathrm{supp}^-(p-q)$, $\lambda = q(v) - p(v)$. Since $p \wedge q + \lambda\chi_v \in [a, b]$, the inductive assumption and (44) imply that

$$g(p) - g(p \wedge q) \geq g(p + \lambda\chi_v) - g((p \wedge q) + \lambda\chi_v) \geq g(p \vee q) - g(q).$$

∎

4.3.2. Fundamental Properties of Polyhedral L-convex Functions

In this section, we show various properties of polyhedral L-convex functions.

LEMMA 4.28. *Let* $g \in \mathcal{L}$. *Then, for any* $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in \mathbf{R}$ *we have*

$$g(p) + g(q) \geq g((p + \lambda\mathbf{1}) \wedge q) + g(p \vee (q - \lambda\mathbf{1})). \tag{46}$$

In particular, we have

$$g(p) + g(q) \geq g(p + \lambda\chi_X) + g(q - \lambda\chi_X)$$

for any $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in [0, \lambda_1 - \lambda_2]$, *where*

$$\lambda_1 = \max_{v\in V}\{q(v) - p(v)\}, \quad X = \{v \in V \mid q(v) - p(v) = \lambda_1\}, \tag{47}$$

$$\lambda_2 = \max_{v\in V\setminus X}\{q(v) - p(v)\}. \tag{48}$$

Proof. The inequality (46) can be obtained as follows:

$$\begin{aligned}
\text{LHS of (46)} &= g(p) + g(q - \lambda\mathbf{1}) + \lambda r \\
&\geq g(p \wedge (q - \lambda\mathbf{1})) + g(p \vee (q - \lambda\mathbf{1})) + \lambda r \\
&= g(\{p \wedge (q - \lambda\mathbf{1})\} + \lambda\mathbf{1}) + g(p \vee (q - \lambda\mathbf{1})) = \text{RHS of (46)}.
\end{aligned}$$

The second inequality follows from this inequality since

$$p \vee \{q - (\lambda_1 - \lambda)\mathbf{1}\} = p + \lambda\chi_X, \qquad (p + (\lambda_1 - \lambda)\mathbf{1}) \wedge q = q - \lambda\chi_X$$

for $\lambda \in [0, \lambda_1 - \lambda_2]$. ▮

Global optimality of a polyhedral L-convex function is characterized by local optimality.

THEOREM 4.29. *Let $g \in \mathcal{L}$ and $p \in \mathrm{dom}\, g$. Then, $g(p) \le g(q)$ $(\forall q \in \mathbf{R}^V)$ $\Longleftrightarrow$ $g'(p;\chi_X) \ge 0$ $(\forall X \subseteq V)$ and $g'(p;\chi_V) = r = 0$, where r is in* (LF2).

Proof. We show the sufficiency by contradiction. Suppose that there exists some $q \in \mathbf{R}^V$ with $g(q) < g(p)$, and assume that q minimizes the number of distinct values in $\{p(v) - q(v)\}_{v \in V}$ of all such vectors. Define λ_1, λ_2, and X $(\subseteq V)$ by (47) and (48). Since $g'(p;\chi_V) = r = 0$, we have $X \neq V$ and $\lambda_2 > -\infty$. By Lemma 4.28, we obtain

$$g(p) + g(q) \ge g(p + (\lambda_1 - \lambda_2)\chi_X) + g(q - (\lambda_1 - \lambda_2)\chi_X). \tag{49}$$

Put $q' = q - (\lambda_1 - \lambda_2)\chi_X$. Since $g'(p;\chi_X) \ge 0$, (49) implies $g(q') \le g(q) < g(p)$. This inequality, however, is a contradiction since the number of distinct values in $\{p(v) - q'(v)\}_{v \in V}$ is less than that of $\{p(v) - q(v)\}_{v \in V}$. ▮

Given a set function $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, define $g_\rho : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$g_\rho(p) = \sum_{j=1}^{k-1} (p_j - p_{j+1})\rho(V_j) + p_k \rho(V_k), \tag{50}$$

where $p_1 > p_2 > \cdots > p_k$ are distinct values in $\{p(v)\}_{v \in V}$, and $V_j = \{v \in V \mid p(v) \ge p_j\}$ $(j = 1, \cdots, k)$. The function g_ρ is called the *Lovász extension* of ρ.

THEOREM 4.30. *If $\rho \in \mathcal{S}$, then*

$$g_\rho(p) = \sup_{x \in \mathrm{B}(\rho)} \langle p, x \rangle \qquad (\forall p \in \mathbf{R}^V).$$

Proof. See [17]. ▮

THEOREM 4.31 (Lovász [24]). *Let $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$. Then, $\rho \in \mathcal{S} \iff g_\rho$ is convex.*

Directional derivative functions and subdifferentials of a polyhedral L-convex function have nice structures such as M/L-convexity, and they can be explicitly described by certain submodular functions (cf. Theorem 4.32 (i)).

THEOREM 4.32. *Let $g \in \mathcal{L}$ and $p \in \mathrm{dom}\, g$.*
(i) *The function $\rho_p : 2^V \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$\rho_p(X) = g'(p; \chi_X) \qquad (X \subseteq V) \tag{51}$$

satisfies $\rho_p(\emptyset) = 0$, $-\infty < \rho_p(V) < +\infty$, and the submodular inequality (7), i.e., $\rho_p \in \mathcal{S}$.
(ii) *We have $g'(p; \cdot) = g_{\rho_p}$ and $g'(p; \cdot) \in {}_0\mathcal{L}$.*

Proof. We show $g'(p; \cdot) \in {}_0\mathcal{L}$ only. The claim (i) is an immediate corollary of (ii), and the equation $g'(p; \cdot) = g_{\rho_p}$ will be shown later in Theorem 4.36 (i).
Let $p \in \mathrm{dom}\, g$. For $q_1, q_2 \in \mathbf{R}^V$ and a sufficiently small $\mu > 0$, we have

$$\begin{aligned} & g'(p; q_1) + g'(p; q_2) \\ &= \{g(p + \mu q_1) - g(p)\}/\mu + \{g(p + \mu q_2) - g(p)\}/\mu \\ &\geq \{g(p + \mu(q_1 \wedge q_2)) - g(p)\}/\mu + \{g(p + \mu(q_1 \vee q_2)) - g(p)\}/\mu \\ &= g'(p; q_1 \wedge q_2) + g'(p; q_1 \vee q_2). \end{aligned}$$

Hence, (LF1) holds for $g'(p; \cdot)$. (LF2) for $g'(p; \cdot)$ can be shown similarly. ▌

M-convexity appears in subdifferentials of a polyhedral L-convex function.

THEOREM 4.33. *Let $g \in \mathcal{L}$ and $p \in \mathrm{dom}\, g$.*
(i) *$\partial g(p) \in \mathcal{M}_0$ and $\partial g(p)$ is represented as*

$$\begin{aligned} \partial g(p) &= \mathrm{B}(\rho_p) \\ &= \{x \in \mathbf{R}^V \mid x(X) \leq g'(p; \chi_X) (\forall X \subseteq V), x(V) = g'(p; \chi_V)\}, \end{aligned} \tag{52}$$

where $\rho_p : 2^V \to \mathbf{R} \cup \{+\infty\}$ *is defined by (51).*
(ii) *For any* $q \in \mathbf{R}^V$ *we have*

$$g(p+q) - g(p) \geq \sup_{x \in \partial g(p)} \langle q, x \rangle = g_{\rho_p}(q).$$

Proof. From Theorems 4.5 and 4.29 follows (52), which, together with Theorem 4.32 (i), implies $\partial g(p) \in \mathcal{M}_0$. The claim (ii) is immediate from (i) and Theorem 4.30. ▮

The next theorem shows that each face of the epigraph of a polyhedral L-convex function is an L-convex polyhedron when it is projected to $\mathbf{R}^V$. The proof is obvious and therefore omitted.

THEOREM 4.34. *For* $g \in \mathcal{L}$ *and* $x \in \mathbf{R}^V$, *we have* $\arg\min g[-x] \in \mathcal{L}_0$ *if* $\inf g[-x] > -\infty$.

The class of polyhedral L-convex functions is closed under various fundamental operations.

THEOREM 4.35. *Let* $g, g_1, g_2 \in \mathcal{L}$.
(1) *For* $x \in \mathbf{R}^V$, *the function* $g[-x] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *given by (25) is polyhedral L-convex.*
(2) *For* $a \in \mathbf{R}^V$, $\beta \in \mathbf{R}$ *and* $\nu > 0$, *the function* $\nu \cdot g(a + \beta p)$ *is polyhedral L-convex in* p.
(3) *For any* $U \subseteq V$, *the function* $g^U : \mathbf{R}^U \to \mathbf{R} \cup \{\pm\infty\}$ *given by (40) is polyhedral L-convex if* $g^U(p_0)$ *is finite for some* $p_0 \in \mathbf{R}^V$.
(4) *For* $\psi_v \in \mathcal{C}^1$ $(v \in V)$, *the function* $\tilde{g} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ *defined by*

$$\tilde{g}(p) = \inf_{q \in \mathbf{R}^V} \{g(q) + \sum_{v \in V} \psi_v(p(v) - q(v))\} \quad (p \in \mathbf{R}^V)$$

is polyhedral L-convex if $\tilde{g}(p_0)$ *is finite for some* $p_0 \in \mathbf{R}^V$.
(5) *For any* $\gamma \in \mathcal{T}$, *the restriction* $g_{\mathrm{D}(\gamma)} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *of* g *defined by*

$$g_{\mathrm{D}(\gamma)}(p) = \begin{cases} g(p) & (p \in \mathrm{D}(\gamma)), \\ +\infty & (p \notin \mathrm{D}(\gamma)). \end{cases}$$

is polyhedral L-convex if $\mathrm{dom}\, g \cap \mathrm{D}(\gamma) \neq \emptyset$.
(6) $g_1 + g_2 \in \mathcal{L}$ *if* $\mathrm{dom}\, g_1 \cap \mathrm{dom}\, g_2 \neq \emptyset$.

A more general and stronger transformation, called "network induction," is explained in Section 7.3.

4.3.3. Positively Homogeneous Polyhedral L-convex Functions

This section clarifies the relationship of positively homogeneous polyhedral L-convex functions with submodular functions, and with M-convex polyhedra.

For a positively homogeneous polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathbf{0} \in \mathrm{dom}\, g$, define a set function $\rho_g : 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho_g(X) = g'(\mathbf{0}; \chi_X) \ (= g(\chi_X)) \qquad (X \subseteq V). \tag{53}$$

Recall the definition of the Lovász extension g_ρ in (50).

THEOREM 4.36.
(i) *For $g \in {}_0\mathcal{L}$, we have $\rho_g \in \mathcal{S}$ and $g_{\rho_g} = g$.*
(ii) *For $\rho \in \mathcal{S}$, we have $g_\rho \in {}_0\mathcal{L}$ and $\rho_{g_\rho} = \rho$.*
(iii) *The mappings $g \mapsto \rho_g$ ($g \in {}_0\mathcal{L}$) and $\rho \mapsto g_\rho$ ($\rho \in \mathcal{S}$) provide a one-to-one correspondence between ${}_0\mathcal{L}$ and $\mathcal{S}$, and are the inverse of each other.*

Proof. (i): (LF1) for g yields $\rho_g \in \mathcal{S}$. We see from Theorem 4.33 (ii) that $g \geq g_{\rho_g}$. On the other hand, from Theorem 4.2 follows

$$\begin{aligned} g_{\rho_g}(p) &= \sum_{j=1}^{k-1}(p_j - p_{j+1})g(\chi_{V_j}) + p_k g(\chi_{V_k}) \\ &\geq g(\sum_{j=1}^{k-1}(p_j - p_{j+1})\chi_{V_j} + p_k\chi_{V_k}) \ = \ g(p), \end{aligned}$$

where p_j and V_j ($j = 1, \cdots, k$) are defined as in the Lovász extension (50).

(ii): We show (LF1) for g_ρ only. The other claims are obvious.

First assume that $\rho < +\infty$. By Theorem 4.27, it suffices to show that (45) holds for any $p \in \mathbf{R}^V$ and $u, v \in V$ ($u \neq v$), where $g = g_\rho$. Define p_j and V_j ($j = 1, \cdots, k$) as in the Lovász extension (50). Let l, m be such that $p(u) = p_l$, $p(v) = p_m$, and $\lambda \in [0, p_{l-1} - p_l]$, $\mu \in [0, p_{m-1} - p_m]$. From the definition of g_ρ we have

$$\begin{aligned} g_\rho(p + \lambda\chi_u) &= g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\}, \\ g_\rho(p + \mu\chi_v) &= g_\rho(p) + \mu\{\rho(V_{m-1} \cup \{v\}) - \rho(V_{m-1})\}. \end{aligned}$$

(Case 1: $l \neq m$): We may assume that $l < m$. Then, (45) holds with equality:

$$\begin{aligned} & g_\rho(p + \lambda\chi_u + \mu\chi_v) \\ &= g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\} + \mu\{\rho(V_{m-1} \cup \{v\}) - \rho(V_{m-1})\} \\ &= g_\rho(p + \lambda\chi_u) + g_\rho(p + \mu\chi_v) - g_\rho(p). \end{aligned}$$

(Case 2: $l = m$): We may assume that $\lambda \geq \mu$. Then,

$$\begin{aligned} & g_\rho(p + \lambda\chi_u + \mu\chi_v) \\ &= g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\} \\ &\qquad + \mu\{\rho(V_{l-1} \cup \{u, v\}) - \rho(V_{l-1} \cup \{u\})\} \\ &\leq g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\} + \mu\{\rho(V_{l-1} \cup \{v\}) - \rho(V_{l-1})\} \\ &= g_\rho(p + \lambda\chi_u) + g_\rho(p + \mu\chi_v) - g_\rho(p), \end{aligned}$$

where the inequality is by the submodularity of ρ. Hence, (45) follows.

Next, we consider the general case where ρ may take the value $+\infty$. Let $x \in \mathrm{B}(\rho)$. For each positive integer k, we define $\rho_k : 2^V \to \mathbf{R}$ by

$$\rho_k(X) = \min_{Y \subseteq X}\{\rho(X \setminus Y) + x(Y) + k|Y|\} \quad (X \subseteq V).$$

Note that ρ_k is the submodular function associated with $\{y \in \mathrm{B}(\rho) \mid y(v) \leq x(v) + k \ (v \in V)\}$, which is a bounded M-convex polyhedron. Hence, we have $\rho_k < +\infty$ and g_{ρ_k} satisfies (LF1). Since $\rho(X) = \lim_{k\to\infty} \rho_k(X)$ $(\forall X \subseteq V)$, it holds that $g_\rho(p) = \lim_{k\to\infty} g_{\rho_k}(p)$ $(p \in \mathbf{R}^V)$ and therefore g_ρ also satisfies (LF1).

(iii): Clear from (i) and (ii). ∎

From Theorem 4.36, we see that a polyhedral convex function is positively homogeneous polyhedral L-convex if and only if it is the Lovász extension of a submodular set function.

COROLLARY 4.37. ${}_0\mathcal{L} = \{g_\rho \mid \rho \in \mathcal{S}\}$.

As immediate consequences of Theorems 3.3, 4.3, and 4.36, we obtain a one-to-one correspondence between positively homogeneous polyhedral L-convex functions and M-convex polyhedra. Recall the notation S_g and δ_S^* in (23) and (24).

THEOREM 4.38.
(i) *For* $g \in {}_0\mathcal{L}$, *we have* $S_g \in \mathcal{M}_0$ *and* $\delta_{S_g}^* = g$.
(ii) *For* $B \in \mathcal{M}_0$, *we have* $\delta_B^* \in {}_0\mathcal{L}$ *and* $S_{\delta_B^*} = B$.
(iii) *The mappings* $g \mapsto S_g$ $(g \in {}_0\mathcal{L})$ *and* $B \mapsto \delta_B^*$ $(B \in \mathcal{M}_0)$ *provide a one-to-one correspondence between* ${}_0\mathcal{L}$ *and* $\mathcal{M}_0$, *and are the inverse of each other.*

4.3.4. Polyhedral $L^\natural$*-convex Functions*

Due to the property (LF2), a polyhedral L-convex function loses no information when restricted to a hyperplane $\{p \in \mathbf{R}^V \mid p(v) = 0\}$ for any $v \in V$. We call a polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *L*$^\natural$*-convex* if the function $\tilde{g} : \mathbf{R}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{g}(p_0, p) = g(p - p_0\mathbf{1}) \quad ((p_0, p) \in \mathbf{R}^{\tilde{V}}) \tag{54}$$

is polyhedral L-convex, where $\tilde{V} = \{v_0\} \cup V$. We see that polyhedral L$^\natural$-convex functions are essentially the same as polyhedral L-convex functions, while the class of polyhedral L$^\natural$-convex functions properly contains that of polyhedral L-convex functions. The concept of L$^\natural$-convexity was originally introduced in [18] as a concept for functions defined over the integer lattice.

THEOREM 4.39. *Let* $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, g \neq \emptyset$. *Then,* g *is L*$^\natural$*-convex* $\iff$ g *satisfies (46) for any* $p, q \in \mathrm{dom}\, g$ *and* $\lambda \geq 0$.

Proof. The "only if" part is clear from the definition of L$^\natural$-convex functions and Lemma 4.28. Hence, we show the "if" part only. It suffices to show that $\tilde{g} : \mathbf{R}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by (54) fulfills (LF1). Let $(p_0, p), (q_0, q) \in \mathrm{dom}\, \tilde{g}$, and without loss of generality assume $p_0 \geq q_0$. Then,

$$\begin{aligned}
\tilde{g}(p_0, p) + \tilde{g}(q_0, q) &= g(p - p_0\mathbf{1}) + g(q - q_0\mathbf{1}) \\
&\geq g((p - p_0\mathbf{1} + (p_0 - q_0)\mathbf{1}) \wedge (q - q_0\mathbf{1})) \\
&\qquad + g((p - p_0\mathbf{1}) \vee (q - q_0\mathbf{1} - (p_0 - q_0)\mathbf{1})) \\
&= g((p \wedge q) - q_0\mathbf{1}) + g((p \vee q) - p_0\mathbf{1}) \\
&= \tilde{g}(p_0 \wedge q_0, p \wedge q) + \tilde{g}(p_0 \vee q_0, p \vee q),
\end{aligned}$$

where the inequality is by (46). Hence, (LF1) holds for $\tilde{g}$. ▮

EXAMPLE 4.40 (separable-convex functions). Let $D \subseteq \mathbf{R}^V$ be an L$^\natural$-convex polyhedron. For any convex functions $g_v \in \mathcal{C}^1$ $(v \in V)$, the function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \begin{cases} \displaystyle\sum_{v \in V} g_v(p(v)) & (p \in D), \\ +\infty & (p \notin D) \end{cases}$$

is polyhedral L$^\natural$-convex if $\mathrm{dom}\, g \neq \emptyset$. ▮

From the definition of polyhedral L$^\natural$-convex functions, every property of polyhedral L-convex functions can be restated in terms of polyhedral L$^\natural$-convex functions, and vice versa. We show two properties below, which seem more natural when restated in terms of polyhedral L$^\natural$-convex functions.

THEOREM 4.41. *Let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral L$^\natural$-convex function.*
(i) *For any $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$, the restriction $g_{[a,b]} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of g to $[a, b]$ defined by (37) is polyhedral L$^\natural$-convex if* $\mathrm{dom}\, g \cap [a, b] \neq \emptyset$.
(ii) *For any $U \subseteq V$, the function $g_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ given by (36) is polyhedral L$^\natural$-convex if* $\mathrm{dom}\, g_U \neq \emptyset$.

Proof. (i) is immediate from the translation of Theorem 4.35 (5), and (ii) is a corollary of (i). ▮

Examples 4.22 and 4.40 show that any separable-convex function is polyhedral M$^\natural$-convex as well as polyhedral L$^\natural$-convex. The converse is also true, as shown in Theorem 4.42 below.

THEOREM 4.42 (cf. [38, Theorem 3.17]). *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$. *Then, $f \in \mathcal{M} \cap \mathcal{L}$ if and only if f is separable-convex.*

Proof. Let $f \in \mathcal{M} \cap \mathcal{L}$. Theorems 3.31, 4.7 (ii), and 4.25 (ii) imply that $\mathrm{dom}\, f$ is an interval, i.e., there exist $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a(v) \leq b(v)$ $(v \in V)$ such that $\mathrm{dom}\, f = [a, b]$. Similarly, Theorems 3.31, 4.17, and 4.34 imply that $\arg\min f[-p]$ is an interval for any $p \in \mathbf{R}^V$ with $\inf f[-p] > -\infty$. Hence, for each $v \in V$ there exist a sequence $\{a_i(v)\}_{i=0}^{k_v} \subseteq \mathbf{R} \cup \{\pm\infty\}$ $(k_v \geq 0)$ such that

- $a_0(v) = a(v) < a_1(v) < \cdots < a_{k_v - 1}(v) < a_{k_v}(v) = b(v)$,
- for any $i_v = 1, 2, \cdots, k_v$ $(v \in V)$, the function f is linear over the interval $\{x \in \mathbf{R}^V \mid a_{i_v - 1}(v) \leq x(v) \leq a_{i_v}(v)\ (v \in V)\}$.

Due to these properties, we have $f(x + \alpha\chi_v) - f(x) = f(y + \alpha\chi_v) - f(y)$ for any $x, y \in \mathrm{dom}\, f$ with $x(v) = y(v)$ and any $\alpha \in \mathbf{R}$. For all $v \in V$, put

$$f_v(\alpha) = f(x_0 + (\alpha - x_0(v))\chi_v) - f(x_0) \quad (\alpha \in \mathbf{R}),$$

where $x_0 \in \mathrm{dom}\, f$. Then, we have

$$f(x) = \sum_{v \in V} f_v(x(v)) + f(x_0) \quad (x \in \mathbf{R}^V).$$

Moreover, each f_v is polyhedral convex since f itself is polyhedral convex. Therefore, f is a separable-convex function. ▮

REMARK 4.43. There exists no function which is both polyhedral M-convex and L-convex, i.e., $\mathcal{M} \cap \mathcal{L} = \emptyset$ (see Remark 3.32). ▮

5. CONJUGACY AND CHARACTERIZATIONS

Polyhedral M-convex and L-convex functions are conjugate to each other.

THEOREM 5.1. *For $f \in \mathcal{M}$ and $g \in \mathcal{L}$, we have $f^\bullet \in \mathcal{L}$ and $g^\bullet \in \mathcal{M}$. More specifically, the mappings $f \mapsto f^\bullet$ ($f \in \mathcal{M}$) and $g \mapsto g^\bullet$ ($g \in \mathcal{L}$) provide a one-to-one correspondence between $\mathcal{M}$ and $\mathcal{L}$, and are the inverse of each other.*

Polyhedral M/L-convex functions are characterized by local polyhedral structures such as directional derivative functions, subdifferentials, and the sets of minimizers.

THEOREM 5.2. *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$. *Then,*

(i) $f \in \mathcal{M}$
$\iff$ (ii) $f'(x; \cdot) \in {}_0\mathcal{M}$ $(\forall x \in \mathrm{dom}\, f)$
$\iff$ (iii) $\partial f(x) \in \mathcal{L}_0$ $(\forall x \in \mathrm{dom}\, f)$
$\iff$ (iv) $\arg\min f[-p] \in \mathcal{M}_0$ $(\forall p \in \mathbf{R}^V$ *with* $\inf f[-p] > -\infty)$.

THEOREM 5.3. *Let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, g \neq \emptyset$. *Then,*

(i) $g \in \mathcal{L}$
$\iff$ (ii) $g'(p; \cdot) \in {}_0\mathcal{L}$ $(\forall p \in \mathrm{dom}\, g)$
$\iff$ (iii) $\partial g(p) \in \mathcal{M}_0$ $(\forall p \in \mathrm{dom}\, g)$
$\iff$ (iv) $\arg\min g[-x] \in \mathcal{L}_0$ $(\forall x \in \mathbf{R}^V$ *with* $\inf g[-x] > -\infty)$.

In the following, we first prove Theorem 5.3, then Theorem 5.1, and finally Theorem 5.2.

Proof of Theorem 5.3 The implication (i) $\Rightarrow$ (ii) is by Theorem 4.32 (ii). To show the reverse implication, we assume (ii) for g. For any $p \in \mathrm{dom}\, g$, there exists $\varepsilon > 0$ such that

$$g(q) - g(p) = g'(p; q-p) \quad (\forall q \in \overline{\mathrm{N}_\infty(p,\varepsilon)}).$$

Hence, g satisfies ($\mathrm{LF1_{loc}}$) and ($\mathrm{LF2_{loc}}$), yielding (i) by Theorem 4.26.

The equivalence (ii) $\Longleftrightarrow$ (iii) is by Theorems 4.4 and 4.38. The implication (i) $\Rightarrow$ (iv) follows from Theorem 4.34. We conclude the proof of Theorem 5.3 by showing that (iv) $\Rightarrow$ (ii) holds.

LEMMA 5.4. *Let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be positively homogeneous polyhedral convex with $\mathrm{dom}\, g \neq \emptyset$. Suppose that $\arg\min g[-x]$ is an L-convex cone for any $x \in \mathbf{R}^V$ with $\inf g[-x] > -\infty$. Then, $g \in {}_0\mathcal{L}$.*

Proof. Define $\rho_g : 2^V \to \mathbf{R} \cup \{+\infty\}$ by (53). Since $\mathbf{0} \in \mathrm{dom}\, g$, there exists some $x \in \mathbf{R}^V$ with $\mathbf{0} \in \arg\min g[-x]$. By (LS2) for $\arg\min g[-x]$, we have $\mathbf{1} \in \arg\min g[-x] \subseteq \mathrm{dom}\, g$, i.e., $\rho_g(V) = g(\mathbf{1}) < +\infty$. As is shown below, we have $g = g_{\rho_g}$, where g_{ρ_g} is defined by (50) with $\rho = \rho_g$. Thus, we have $\rho_g \in \mathcal{S}$ by Theorem 4.31, which, together with Corollary 4.37, implies $g \in {}_0\mathcal{L}$.

We now prove $g = g_{\rho_g}$. Note that $g(p) \le g_{\rho_g}(p)$ ($p \in \mathbf{R}^V$) by Theorem 4.2. This implies that $g(p) = g_{\rho_g}(p) = +\infty$ if $p \notin \mathrm{dom}\, g$. Therefore, we may assume that $p \in \mathrm{dom}\, g$. Then, there exists some $x \in \mathbf{R}^V$ such that $p \in \arg\min g[-x]$. By Theorem 3.25, there exists some chain $\mathcal{F}_0$ ($\subseteq 2^V$) and $(\lambda_X \mid X \in \mathcal{F}_0)$ ($\subseteq \mathbf{R}$) such that

$$\begin{aligned} p &= \sum_{X \in \mathcal{F}_0} \lambda_X \chi_X, \quad \lambda_X \ge 0 \ (\forall X \in \mathcal{F}_0 \setminus \{V\}), \\ \mathcal{F}_0 &\subseteq \{X \subseteq V \mid \chi_X \in \arg\min g[-x]\}. \end{aligned}$$

By the linearity of g over $\arg\min g[-x]$ and by (50), we have

$$g(p) = \sum_{X \in \mathcal{F}_0} \lambda_X g(\chi_X) = g_{\rho_g}(p).$$

∎

We now prove (iv) $\Rightarrow$ (ii). For any $p \in \operatorname{dom} g$, we denote $g'_p = g'(p;\cdot)$ for notational simplicity. By Lemma 5.4, we have only to prove that $\arg\min g'_p[-x]$ is an L-convex cone for any $p \in \operatorname{dom} g$ and any $x \in \mathbf{R}^V$ with $\inf g'_p[-x] > -\infty$.

Let $p \in \operatorname{dom} g$, and $x \in \mathbf{R}^V$ be such that $\inf g'_p[-x] > -\infty$. Then, we have $\inf g[-x] > -\infty$ and $\arg\min g[-x] \in \mathcal{L}_0$. Let $\gamma \in \mathcal{T}$ be such that $\mathrm{D}(\gamma) = \arg\min g[-x]$. Since $\arg\min g'_p[-x]$ is the tangent cone of $\arg\min g[-x]$ at p, it can be represented as

$$\arg\min g'_p[-x] = \{q \in \mathbf{R}^V \mid q(v) - q(u) \leq 0 \ (\forall (u,v) \in A_p)\},$$

where $A_p = \{(u,v) \mid u, v \in V,\ p(v) - p(u) = \gamma(u,v)\}$. Since A_p is transitive, $\arg\min g'_p[-x]$ is an L-convex cone by Theorem 3.24. Thus, Theorem 5.3 is proven. ∎

The following lemma is needed for the proof of Theorem 5.1.

Lemma 5.5. *Let $g \in \mathcal{L}$, and $x, y \in \mathbf{R}^V$ be such that* $\inf g[-x] > -\infty$ *and* $\inf g[-y] > -\infty$. *Then, for any $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that*

$$p(v) - p(u) \leq q(v) - q(u) \quad (\forall p \in \arg\min g[-x], \forall q \in \arg\min g[-y]). \quad (55)$$

Proof. First we note that $x(V) = y(V) = r$, where $r \in \mathbf{R}$ is the value in (LF2) for g. Let $u \in \mathrm{supp}^+(x-y)$. We have $\arg\min g[-x], \arg\min g[-y] \in \mathcal{L}_0$ by Theorem 4.34. Hence, it suffices to show the following: there exists some $v \in \mathrm{supp}^-(x-y)$ such that $p(v) \leq q(v)$ $(\forall p \in D_x,\ \forall q \in D_y)$, where

$$\begin{aligned} D_x &= \{p \mid p \in \arg\min g[-x],\ p(u) = 0\}, \\ D_y &= \{q \mid q \in \arg\min g[-y],\ q(u) = 0\}. \end{aligned}$$

Assume, to the contrary, that for any $v \in \mathrm{supp}^-(x-y)$, there exists a pair of vectors $p_v \in D_x$, $q_v \in D_y$ such that $p_v(v) > q_v(v)$. Set

$$p_* = \bigvee \{p_v \mid v \in \mathrm{supp}^-(x-y)\}, \quad q_* = \bigwedge \{q_v \mid v \in \mathrm{supp}^-(x-y)\}.$$

Then, we have $p_* \in D_x$, $q_* \in D_y$, and $p_*(v) > q_*(v)$ $(\forall v \in \mathrm{supp}^-(x-y))$. Let $\lambda > 0$ be any value with $\lambda \leq p_*(v) - q_*(v)$ for all $v \in \mathrm{supp}^+(p_* - q_*)$.

Putting

$$\begin{aligned} p' &= (p_* - \lambda \mathbf{1}) \vee q_* = \begin{cases} p_*(v) - \lambda & (v \in \mathrm{supp}^+(p_* - q_*)), \\ q_*(v) & (v \in V \setminus \mathrm{supp}^+(p_* - q_*)), \end{cases} \\ q' &= p_* \wedge (q_* + \lambda \mathbf{1}) = \begin{cases} q_*(v) + \lambda & (v \in \mathrm{supp}^+(p_* - q_*)), \\ p_*(v) & (v \in V \setminus \mathrm{supp}^+(p_* - q_*)), \end{cases} \end{aligned}$$

we have

$$g(p_*) + g(q_*) \geq g(p') + g(q') \tag{56}$$

by Lemma 4.28. Since $\mathrm{supp}^-(x-y) \subseteq \mathrm{supp}^+(p_* - q_*)$, we obtain

$$\begin{aligned} &\langle p', x\rangle + \langle q', y\rangle - \langle p_*, x\rangle - \langle q_*, y\rangle \\ &\quad = \lambda \sum \{y(v) - x(v) \mid v \in \mathrm{supp}^+(p_* - q_*)\} \\ &\qquad + \sum \big\{(q_*(v) - p_*(v))(x(v) - y(v)) \ \big| \ v \in V \setminus \mathrm{supp}^+(p_* - q_*)\big\} \\ &\quad \geq \lambda \sum \{y(v) - x(v) \mid v \in V \setminus \{u\}\} \ = \ \lambda\{x(u) - y(u)\} \ > \ 0. \end{aligned}$$

Combining this inequality with (56), we have

$$g[-x](p') + g[-y](q') < g[-x](p_*) + g[-y](q_*),$$

which is a contradiction since $p_* \in \arg\min g[-x]$, $q_* \in \arg\min g[-y]$. ▮

Proof of Theorem 5.1 First recall Theorem 4.1.

$[g \in \mathcal{L} \Rightarrow g^\bullet \in \mathcal{M}]$ Let $x, y \in \mathrm{dom}\, g^\bullet$. Then, $\inf g[-x] > -\infty$ and $\inf g[-y] > -\infty$. By Lemma 5.5, for any $u \in \mathrm{supp}^+(x-y)$ there exists $v \in \mathrm{supp}^-(x-y)$ satisfying the inequality (55), which implies

$$\begin{aligned} &(g^\bullet)'(x; v, u) + (g^\bullet)'(y; u, v) \\ &\quad = \ \sup\{p(v) - p(u) \mid p \in \arg\min g[-x]\} \\ &\qquad\qquad + \ \sup\{q(u) - q(v) \mid q \in \arg\min g[-y]\} \ \leq \ 0, \end{aligned}$$

where the equality is by Theorems 4.4 and 4.5. Hence, we have (M-EXC$'$) for $g^\bullet$ and therefore (M-EXC) by Theorem 4.11.

$[f \in \mathcal{M} \Rightarrow f^\bullet \in \mathcal{L}]$ If $f \in \mathcal{M}$, then Theorems 4.5 and 4.17 imply

$$\partial f^\bullet(p) = \arg\min f[-p] \in \mathcal{M}_0 \quad (\forall p \in \mathrm{dom}\, f^\bullet),$$

which, together with Theorem 5.3, yields $f^\bullet \in \mathcal{L}$. ▮

Proof of Theorem 5.2 The equivalence (ii) $\Longleftrightarrow$ (iii) is by Theorems 4.4 and 4.20. By Theorems 4.5 and 5.1, the equivalence (i) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$

(iv) is rewritten as follows in terms of $f^\bullet$:

$$\begin{aligned} f^\bullet \in \mathcal{L} &\iff \arg\min f^\bullet[-x] \in \mathcal{L}_0 \quad (\forall x \in \mathbf{R}^V \text{ with } \inf f^\bullet[-x] > -\infty) \\ &\iff \partial f^\bullet(p) \in \mathcal{M}_0 \ (\forall p \in \operatorname{dom} f^\bullet), \end{aligned}$$

which is already shown in Theorem 5.3. ▮

Proof of Theorem 4.18 (6) From the assumption and Theorem 4.6, $f_1 \Box f_2$ is a polyhedral convex function with $(f_1 \Box f_2)(x) > -\infty$ $(\forall x \in \mathbf{R}^V)$. Hence, it follows from Theorem 4.1 that $(f_1 \Box f_2)^\bullet$ is also a polyhedral convex function such that $(f_1 \Box f_2)^\bullet(x) > -\infty$ $(\forall x \in \mathbf{R}^V)$ and $\operatorname{dom}(f_1 \Box f_2)^\bullet \neq \emptyset$. Theorems 4.6, 4.35 (6), and 5.1 imply $(f_1 \Box f_2)^\bullet = f_1^\bullet + f_2^\bullet \in \mathcal{L}$, which shows $f_1 \Box f_2 \in \mathcal{M}$ by Theorem 5.1. ▮

Proof of Theorem 4.21 Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with $\operatorname{dom} f \neq \emptyset$ satisfying (M$^\natural$-EXC). Let $p \in \mathbf{R}^V$ be any vector with $\inf f[-p] > -\infty$. It follows from (M$^\natural$-EXC) that $\arg\min f[-p]$ satisfies (G-EXC), i.e., $\arg\min f[-p]$ is an M$^\natural$-convex polyhedron by Theorem 3.8. From the definition of polyhedral M$^\natural$-convex functions and Theorem 5.2, the function f is polyhedral M$^\natural$-convex. ▮

6. RELATIONSHIP WITH M/L-CONVEX FUNCTIONS OVER THE INTEGER LATTICE

The concepts of M-convexity and L-convexity were originally introduced for functions defined over the integer lattice [29, 30, 32]. In this section, we present the relationship of M/L-convexity over the integer lattice and polyhedral M/L-convexity.

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *discrete M-convex* if $\operatorname{dom}_{\mathbf{Z}} f \neq \emptyset$ and it satisfies

(M-EXC[Z]) $\forall x, y \in \operatorname{dom}_{\mathbf{Z}} f$, $\forall u \in \operatorname{supp}^+(x-y)$, $\exists v \in \operatorname{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\operatorname{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}$. On the other hand, a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *discrete L-convex* if $\operatorname{dom}_{\mathbf{Z}} g \neq \emptyset$ and it satisfies

(LF1[Z]) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \operatorname{dom}_{\mathbf{Z}} g)$,
(LF2[Z]) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r$ $(\forall p \in \operatorname{dom}_{\mathbf{Z}} g,\ \forall \lambda \in \mathbf{Z})$.

We denote by $\mathcal{M}[\mathbf{Z}]$ and $\mathcal{L}[\mathbf{Z}]$, respectively, the classes of discrete M-convex and L-convex functions:

$$\begin{aligned}\mathcal{M}[\mathbf{Z}] &= \{f \mid f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}, \text{ discrete M-convex}\},\\ \mathcal{L}[\mathbf{Z}] &= \{g \mid g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}, \text{ discrete L-convex}\}.\end{aligned}$$

In the following, we sometimes regard $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ as a function defined over $\mathbf{R}^V$. In such a case, we have $\operatorname{dom} f \subseteq \mathbf{Z}^V$.

REMARK 6.1. The concept of M-convex function over the integer lattice was first defined in [29, 30], where the effective domain is assumed to be bounded. The paper [32] discusses the case when the effective domain is not necessarily bounded, but assumes that functions take only integer values.

The concept of L-convex function over the integer lattice appeared first in [32], where function values are assumed be integral. Hence, the definition of L-convex functions above is a slight extension of the original one. ▌

We also define the set version of discrete M/L-convexity as follows. A set $B \subseteq \mathbf{Z}^V$ is called *discrete M-convex* if $B \neq \emptyset$ and it satisfies

(B-EXC[Z]) $\forall x, y \in B$, $\forall u \in \operatorname{supp}^+(x-y)$, $\exists v \in \operatorname{supp}^-(x-y)$ such that

$$x - \chi_u + \chi_v \in B, \quad y + \chi_u - \chi_v \in B.$$

We call $D \subseteq \mathbf{Z}^V$ *discrete L-convex* if $D \neq \emptyset$ and it satisfies

(LS1[Z]) $p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D$,
(LS2[Z]) $p \in D \Longrightarrow p + \lambda\mathbf{1} \in D \quad (\forall \lambda \in \mathbf{Z})$.

We denote

$$\begin{aligned}\mathcal{M}_0[\mathbf{Z}] &= \{B \mid B \subseteq \mathbf{Z}^V, \text{ discrete M-convex}\},\\ \mathcal{L}_0[\mathbf{Z}] &= \{D \mid D \subseteq \mathbf{Z}^V, \text{ discrete L-convex}\}.\end{aligned}$$

In this section, we use some known results on discrete M/L-convexity. See [29, 30, 32, 35] as references.

Discrete M/L-convex functions can be extended to ordinary convex functions by taking their convex closure. Moreover, convex extensions of discrete M/L-convex functions coincide with "local" convex extensions. For any $x \in \mathbf{R}^V$ and any integer k (≥ 0), we define the set

$$\mathrm{HC}_k(x) = \{y \in \mathbf{Z}^V \mid \lfloor x(v) \rfloor - k \leq y(v) \leq \lceil x(v) \rceil + k \ (v \in V)\}.$$

THEOREM 6.2. *For $B \in \mathcal{M}_0[\mathbf{Z}]$, we have $\overline{B} \in \mathcal{M}_0$ and*

$$\overline{B \cap \mathrm{HC}_0(x)} = \{y \in \overline{B} \mid \lfloor x(v) \rfloor \le y(v) \le \lceil x(v) \rceil (v \in V)\} \quad (x \in \mathbf{R}^V). \quad (57)$$

Proof. Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ be the submodular function such that $B = \mathrm{B}(\rho) \cap \mathbf{Z}^V$. Since $\mathrm{B}(\rho)$ is an integral polyhedron (cf. [17]), we have $\mathrm{B}(\rho) = \overline{B} \in \mathcal{M}_0$. For any $a, b \in \mathbf{Z}^V$ the set $\overline{B} \cap [a, b]$ is also an integral polyhedron if it is nonempty. Thus (57) follows. ▮

THEOREM 6.3. *For $D \in \mathcal{L}_0[\mathbf{Z}]$, we have $\overline{D} \in \mathcal{L}_0$ and*

$$\overline{D \cap \mathrm{HC}_0(p)} = \{q \in \overline{D} \mid \lfloor p(v) \rfloor \le q(v) \le \lceil p(v) \rceil \ (v \in V)\} \quad (p \in \mathbf{R}^V). \quad (58)$$

Proof. Let $\gamma : V \times V \to \mathbf{Z} \cup \{+\infty\}$ be the distance function with triangle inequality which is associated with D (see [32, Theorem 4.20], [35, Theorem 2.28]), i.e., γ is such that $D = \mathrm{D}(\gamma) \cap \mathbf{Z}^V$. Lemma 3.21 (v) implies $\overline{D} = \mathrm{D}(\gamma) \in \mathcal{L}_0$. Note that for any $a, b \in \mathbf{Z}^V$ the set $\overline{D} \cap [a, b]$ is also an integral polyhedron if it is nonempty. Thus (58) follows. ▮

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. The *convex closure* $\bar{f} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of f is given by

$$\bar{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \le f(y) \ (y \in \mathbf{Z}^V)\} \ (x \in \mathbf{R}^V). \quad (59)$$

We also define a function $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \le f(y) \ (y \in \mathrm{HC}_0(x))\} \quad (x \in \mathbf{R}^V). \quad (60)$$

Note that $\tilde{f}$ is the local convex extension of f, i.e., the convex closure of the restriction of f to the integral points around x. It admits an alternative expression

$$\tilde{f}(x) = \inf \left\{ \sum_{y \in \mathrm{HC}_0(x)} \lambda_y f(y) \;\middle|\; \begin{array}{l} \sum_{y \in \mathrm{HC}_0(x)} \lambda_y y = x, \ \sum_{y \in \mathrm{HC}_0(x)} \lambda_y = 1, \\ \lambda_y \ge 0 \ (y \in \mathrm{HC}_0(x)) \end{array} \right\} \quad (x \in \mathbf{R}^V) \quad (61)$$

by LP duality. From the definition, we have

$$\tilde{f}(x) \geq \bar{f}(x) \ (\forall x \in \mathbf{R}^V), \qquad \tilde{f}(x) = f(x) \ (\forall x \in \mathbf{Z}^V). \tag{62}$$

Favati–Tardella [11] investigated the class of functions $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ for which $\bar{f} = \tilde{f}$ holds. They call such a function *integrally convex*. It is easy to see that global optimality of integrally convex functions can be characterized by local optimality.

THEOREM 6.4 ([11, Proposition 3.1]). *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an integrally convex function and $x \in \operatorname{dom} f$. Then, $f(x) \leq f(y)$ for any $y \in \mathbf{Z}^V$ if and only if $f(x) \leq f(y)$ for any $y \in \mathbf{Z}^V$ with $||y - x||_\infty \leq 1$.* ▮

Theorems 6.5 and 6.7 show that discrete M/L-convex functions are integrally convex.

THEOREM 6.5. *For $f \in \mathcal{M}[\mathbf{Z}]$, we have $\bar{f}(x) = \tilde{f}(x)$ $(\forall x \in \mathbf{R}^V)$ and $\bar{f}(x) = f(x)$ $(\forall x \in \mathbf{Z}^V)$.*

Proof. We prove the former only. The latter follows from the former and the equation in (62).

We first consider the case when $x \notin \overline{\operatorname{dom}_{\mathbf{Z}} f}$. Define the conjugate $f^\bullet$ of f by (22). Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ be the submodular function associated with the discrete M-convex set $\operatorname{dom}_{\mathbf{Z}} f \in \mathcal{M}_0[\mathbf{Z}]$, and $p \in \operatorname{dom} f^\bullet$. If $x(V) \neq \rho(V)$, then

$$\begin{aligned} \bar{f}(x) &\geq \sup_{\alpha \in \mathbf{R}} \left[\langle p + \alpha\chi_V, x\rangle - \sup_{y \in \mathbf{R}^V} \{\langle p + \alpha\chi_V, y\rangle - f(y)\} \right] \\ &= \langle p, x\rangle - f^\bullet(p) + \sup_{\alpha \in \mathbf{R}} \alpha\{x(V) - \rho(V)\} \ = \ +\infty. \end{aligned}$$

In the similar way, we can show that $\bar{f}(x) = +\infty$ when $x(X) > \rho(X)$ $(\exists X \subset V)$. By the inequality in (62), we have $\tilde{f}(x) = \bar{f}(x) = +\infty$

To show $\bar{f}(x) = \tilde{f}(x)$ for $x \in \overline{\operatorname{dom}_{\mathbf{Z}} f}$, we consider the following dual pair of LP problems:

(LP1) Maximize $\langle p, x\rangle + \alpha$
subject to $\langle p, y\rangle + \alpha \le f(y)\ (y \in \mathrm{HC}_1(x)),\ p \in \mathbf{R}^V,\ \alpha \in \mathbf{R}$,

(LP2) Minimize $\displaystyle\sum_{y\in \mathrm{HC}_1(x)} \lambda_y f(y)$
subject to $\displaystyle\sum_{y\in \mathrm{HC}_1(x)} \lambda_y y = x,\quad \sum_{y\in \mathrm{HC}_1(x)} \lambda_y = 1,\ \lambda_y \ge 0\ (y \in \mathrm{HC}_1(x))$.

(LP2) has a feasible solution since $x \in \overline{\mathrm{HC}_1(x) \cap \mathrm{dom}_{\mathbf{Z}} f}$. Let (p^*, α^*) $(\in \mathbf{R}^V \times \mathbf{R})$ and $\lambda^* = (\lambda^*_y \mid y \in \mathrm{HC}_1(x))$ be optimal solutions of (LP1) and (LP2), respectively. Then, it holds that

$$\bar{f}(x) \le \langle p^*, x\rangle + \alpha^* = \sum_{y\in \mathrm{HC}_1(x)} \lambda^*_y f(y) \le \tilde{f}(x). \tag{63}$$

We will show that both of the inequalities hold with equality.

Put

$$B = \{y \in \mathrm{HC}_1(x) \mid \langle p^*, y\rangle + \alpha^* = f(y)\} = \arg\min_{y\in \mathrm{HC}_1(x)} f[-p^*](y) \quad (\in \mathcal{M}_0[\mathbf{Z}]).$$

The complementary slackness condition implies $\{y \in \mathrm{HC}_1(x) \mid \lambda^*_y > 0\} \subseteq B$, which shows $x \in \overline{B}$. In particular, we have $x \in \overline{B \cap \mathrm{HC}_0(x)}$ by Theorem 6.2 since $B \in \mathcal{M}_0[\mathbf{Z}]$. Hence, there is another optimal solution $\tilde{\lambda} = (\tilde{\lambda}_y \mid y \in \mathrm{HC}_1(x))$ of (LP2) such that if $\tilde{\lambda}_y > 0$ then $y \in B \cap \mathrm{HC}_0(x)$. Since

$$\sum_{y\in \mathrm{HC}_1(x)} \lambda^*_y f(y) = \sum_{y\in \mathrm{HC}_1(x)} \tilde{\lambda}_y f(y) = \sum_{y\in \mathrm{HC}_0(x)} \tilde{\lambda}_y f(y) \ge \tilde{f}(x),$$

the second inequality in (63) holds with equality.

Let $y_0 \in B \cap \mathrm{HC}_0(x)$. Then, we have $f[-p^*](y_0 - \chi_u + \chi_v) \ge f[-p^*](y_0)$ for any $u, v \in V$. Since local optimality means global optimality for discrete M-convex functions [32, Theorem 4.6], we have $\alpha^* = f[-p^*](y_0) \le f[-p^*](y) = -\langle p^*, y\rangle + f(y)$ $(\forall y \in \mathrm{dom}_{\mathbf{Z}} f)$, i.e., $\langle p^*, y\rangle + \alpha^* \le f(y)$ $(\forall y \in \mathrm{dom}_{\mathbf{Z}} f)$. By (59), the first inequality in (63) holds with equality. ▮

The following theorem claims that we can choose a common optimal λ in (61) for two discrete M-convex functions. An implication of this fact is discussed later in Corollary 6.8 (i).

THEOREM 6.6. *For any* $f, g \in \mathcal{M}[\mathbf{Z}]$ *and* $x \in \mathbf{R}^V$, *there exists* $\lambda = (\lambda_y \mid y \in \mathrm{HC}_0(x))$ *such that*

$$\sum_{y \in \mathrm{HC}_0(x)} \lambda_y y = x, \qquad \sum_{y \in \mathrm{HC}_0(x)} \lambda_y = 1, \quad \lambda_y \geq 0 \ (y \in \mathrm{HC}_0(x)), \tag{64}$$

$$\bar{f}(x) = \tilde{f}(x) = \sum_{y \in \mathrm{HC}_0(x)} \lambda_y f(y), \quad \bar{g}(x) = \tilde{g}(x) = \sum_{y \in \mathrm{HC}_0(x)} \lambda_y g(y). \tag{65}$$

Proof. We may assume that $x \in \overline{\mathrm{dom}_{\mathbf{Z}} f} \cap \overline{\mathrm{dom}_{\mathbf{Z}} g}$, which implies both $\tilde{f}(x)$ and $\tilde{g}(x)$ are finite. By (60), there exist $(p, \alpha), (q, \beta) \in \mathbf{R}^V \times \mathbf{R}$ such that

$$\begin{array}{l} \langle p, y \rangle + \alpha \leq f(y) \ (y \in \mathrm{HC}_0(x)), \ \langle p, x \rangle + \alpha = \tilde{f}(x), \\ \langle q, y \rangle + \beta \leq g(y) \ (y \in \mathrm{HC}_0(x)), \ \langle q, x \rangle + \beta = \tilde{g}(x). \end{array}$$

Put

$$\begin{array}{rcl} B_f &=& \{y \in \mathrm{HC}_0(x) \mid \langle p, y \rangle + \alpha = f(y)\} \ (\in \mathcal{M}_0[\mathbf{Z}]), \\ B_g &=& \{y \in \mathrm{HC}_0(x) \mid \langle q, y \rangle + \beta = g(y)\} \ (\in \mathcal{M}_0[\mathbf{Z}]). \end{array}$$

Then, we have $x \in \overline{B_f} \cap \overline{B_g} = \overline{B_f \cap B_g}$. Therefore, there exists $\lambda = (\lambda_y \mid y \in \mathrm{HC}_0(x))$ satisfying (64) and $\lambda_y = 0$ $(y \notin B_f \cap B_g)$. Such λ satisfies (65) by LP-duality. ▮

THEOREM 6.7 ([32, Theorem 4.18]). *Let* $g \in \mathcal{L}[\mathbf{Z}]$.
(1) *For* $q \in \mathrm{dom}_{\mathbf{Z}}\, g$ *and* $p \in [0,1]^V$, *we have*

$$\begin{array}{rcl} \bar{g}(q+p) &=& \tilde{g}(q+p) \\ &=& g(q) + \displaystyle\sum_{j=1}^{k} (p_j - p_{j+1}) \{g(q + \chi_{V_j}) - g(q)\}, \end{array} \tag{66}$$

where p_j *and* V_j $(j = 1, \cdots, k)$ *are as in the definition of the Lovász extension (50) and* $p_{k+1} = 0$.
(2) $\bar{g}(p) = \tilde{g}(p) = g(p)$ $(\forall p \in \mathbf{Z}^V)$.
(3) $\bar{g}(p + \lambda \mathbf{1}) = \bar{g}(p) + \lambda r$ $(\forall p \in \mathbf{R}^V, \ \lambda \in \mathbf{R})$, *where* r *is in* (LF2[**Z**]).
(4) *The equalities in (66) remain valid for* $q \in \mathrm{dom}_{\mathbf{Z}}\, g$ *and* $p \in \mathbf{R}^V$ *with* $\max_{v \in V} p(v) - \min_{v \in V} p(v) \leq 1$.

As a corollary of Theorems 6.6 and 6.7, we obtain the following properties. For a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, we define the *concave closure*

$\widehat{g} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of g by

$$\widehat{g}(x) = \inf_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x\rangle + \alpha \mid \langle p, y\rangle + \alpha \geq g(y) \ (y \in \mathbf{Z}^V)\} \quad (x \in \mathbf{R}^V).$$

Note that $\widehat{g} = -\overline{(-g)}$.

COROLLARY 6.8.
(i) *Let f, g be functions such that $f, -g \in \mathcal{M}[\mathbf{Z}]$. If $f(x) \geq g(x)$ for any $x \in \mathbf{Z}^V$, then we have $\bar{f}(x) \geq \widehat{g}(x)$ for any $x \in \mathbf{R}^V$.*
(ii) *Let f, g be functions such that $f, -g \in \mathcal{L}[\mathbf{Z}]$. If $f(p) \geq g(p)$ for any $p \in \mathbf{Z}^V$, then we have $\bar{f}(p) \geq \widehat{g}(p)$ for any $p \in \mathbf{R}^V$.*

The following theorems show that the convex extension of discrete M/L-convex functions are closely related to polyhedral M/L-convexity.

THEOREM 6.9. *Let $f \in \mathcal{M}[\mathbf{Z}]$. If $\bar{f}$ is polyhedral convex, then $\bar{f} \in \mathcal{M}$. In particular, $\bar{f} \in \mathcal{M}$ if $\mathrm{dom}_{\mathbf{Z}} f$ is bounded.*

Proof. By Theorems 6.2 and 6.5 (see also [32, Theorem 4.11]), we have

$$\arg\min \bar{f}[-p] = \overline{\arg\min f[-p]} \in \mathcal{M}_0 \qquad (\forall p \in \mathbf{R}^V),$$

which, together with Theorem 5.2, implies $\bar{f} \in \mathcal{M}$. ▮

THEOREM 6.10. *Let $g \in \mathcal{L}[\mathbf{Z}]$. If $\bar{g}$ is polyhedral convex, then $\bar{g} \in \mathcal{L}$. In particular, $\bar{g} \in \mathcal{L}$ if $\{p \in \mathrm{dom}_{\mathbf{Z}}\, g \mid p(v) = 0\}$ is bounded for some $v \in V$.*

Proof. By Theorems 6.3 and 6.7 (1), we have

$$\arg\min \bar{g}[-x] = \overline{\arg\min g[-x]} \in \mathcal{L}_0 \qquad (\forall x \in \mathbf{R}^V),$$

which, together with Theorem 5.3, implies $\bar{g} \in \mathcal{L}$. ▮

REMARK 6.11. For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, if $\mathrm{dom}_{\mathbf{Z}} f$ is bounded then its convex closure $\bar{f}$ is polyhedral convex, but in general $\bar{f}$ is not necessarily polyhedral convex, even if f is discrete M/L-convex.

For example, consider the function $f : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x, y) = \begin{cases} x^2 & ((x, y) \in \mathbf{Z}^2,\ x + y = 0), \\ +\infty & (\text{otherwise}). \end{cases}$$

It is easy to see that f is discrete M-convex. However, the convex closure $\bar{f}$ of f has infinite number of linear-pieces, and is not a polyhedral convex function. ▮

We now introduce two new classes of M/L-convexity. We call a function *integral M-convex* (resp. *integral L-convex*) if it is represented as the convex closure of some integer-valued discrete M-convex (resp. discrete L-convex) function. We denote by $\mathcal{M}_{\rm int}$ and $\mathcal{L}_{\rm int}$, respectively, the classes of integral M-convex and L-convex functions:

$$\begin{aligned} \mathcal{M}_{\rm int} &= \{f \mid f = \overline{f_{\mathbf{Z}}} \text{ for some } f_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\},\ f_{\mathbf{Z}} \in \mathcal{M}[\mathbf{Z}]\}, \\ \mathcal{L}_{\rm int} &= \{g \mid g = \overline{g_{\mathbf{Z}}} \text{ for some } g_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\},\ g_{\mathbf{Z}} \in \mathcal{L}[\mathbf{Z}]\}. \end{aligned}$$

Note that integral M/L-convex functions may have infinite number of linear pieces and therefore are not polyhedral convex in general if the effective domain is unbounded (see Remark 6.11).

It was shown in [32] that integer-valued discrete M-convex and L-convex functions are conjugate to each other under the integer Fenchel transformation.

THEOREM 6.12 ([32, Theorem 4.24]).
(i) *For an integer-valued discrete M-convex function* $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, *the function* $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *defined by*

$$g(p) = \sup_{x \in \mathbf{Z}^V} \{\langle p, x\rangle - f(x)\} \quad (p \in \mathbf{Z}^V)$$

is integer-valued discrete L-convex.
(ii) *For an integer-valued discrete L-convex function* $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, *the function* $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *defined by*

$$f(x) = \sup_{p \in \mathbf{Z}^V} \{\langle p, x\rangle - g(p)\} \quad (x \in \mathbf{Z}^V)$$

is integer-valued discrete M-convex.
(iii) *The mappings* $f \mapsto g$ *and* $g \mapsto f$ *defined in* (i) *and* (ii), *respectively, provide a one-to-one correspondence between the classes of integer-valued discrete M-convex and L-convex functions, and are the inverse of each other.*

Using this conjugacy relationship, we can show that integral M-convex and L-convex functions are conjugate to each other.

LEMMA 6.13. *Let* $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *be an integrally convex function. For any* $p \in \mathbf{R}^V$, *if* $\inf f[-p]$ *is finite, then* $\arg\min f[-p] \neq \emptyset$.

Proof. Put

$$\varepsilon_0 = \min_{S,T \subseteq V} \{p(S) - p(T) - \lceil (p(S) - p(T)) \rceil + 1\} \ (> 0).$$

We claim that for any $x, y \in \mathrm{dom}_{\mathbf{Z}} f$ with $\|y - x\|_\infty \leq 1$, if $f[-p](y) < f[-p](x)$ then $f[-p](y) \leq f[-p](x) - \varepsilon_0$. Since $\|y - x\|_\infty \leq 1$, there exist some $S, T \subseteq V$ such that $y = x + \chi_S - \chi_T$. If $f[-p](y) < f[-p](x)$ then we have $f(y) - f(x) < p(S) - p(T) \leq \lceil (p(S) - p(T)) \rceil$, implying $f(y) - f(x) \leq \lceil (p(S) - p(T)) \rceil - 1$ since $f(y) - f(x) \in \mathbf{Z}$. Hence, we have the claim.

Let $x_* \in \mathrm{dom}_{\mathbf{Z}} f$ satisfy $f[-p](x_*) \leq \inf f[-p] + \varepsilon_0/2$, and $y \in \mathrm{dom}_{\mathbf{Z}} f$ be any vector with $\|y - x_*\|_\infty \leq 1$. If $f[-p](y) < f[-p](x_*)$, then

$$-\varepsilon_0 \geq f[-p](y) - f[-p](x_*) \geq \inf f[-p] - (\inf f[-p] + \varepsilon_0/2) \geq -\varepsilon_0/2,$$

a contradiction. Hence, we have $f[-p](y) \geq f[-p](x_*)$ for any $y \in \mathrm{dom}_{\mathbf{Z}} f$ with $\|y - x_*\|_\infty \leq 1$. By Theorem 6.4 we have $x_* \in \arg\min f[-p]$. ▮

For any $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$, we define $f_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_{\mathbf{Z}}(x) = \begin{cases} f(x) & (x \in \mathbf{Z}^V), \\ +\infty & (x \notin \mathbf{Z}^V). \end{cases} \tag{67}$$

THEOREM 6.14.
(i) *For* $f \in \mathcal{M}_{\mathrm{int}}$, *it holds that* $f^\bullet \in \mathcal{L}_{\mathrm{int}}$ *and*

$$f^\bullet(p) = \sup_{x \in \mathbf{Z}^V} \{\langle p, x \rangle - f(x)\} \qquad (p \in \mathbf{R}^V). \tag{68}$$

In particular, for any $p \in \mathrm{dom}\, f^\bullet$ *there exists some* $x \in \mathrm{dom}\, f \cap \mathbf{Z}^V$ *such that* $f^\bullet(p) = \langle p, x \rangle - f(x)$.
(ii) *For* $g \in \mathcal{L}_{\mathrm{int}}$, *it holds that* $g^\bullet \in \mathcal{M}_{\mathrm{int}}$ *and*

$$g^\bullet(x) = \sup_{p \in \mathbf{Z}^V} \{\langle p, x \rangle - g(p)\} \qquad (x \in \mathbf{R}^V).$$

In particular, for any $x \in \mathrm{dom}\, g^\bullet$ *there exists some* $p \in \mathrm{dom}\, g \cap \mathbf{Z}^V$ *such that* $g^\bullet(x) = \langle p, x \rangle - g(p)$.
(iii) *The mappings* $f \mapsto f^\bullet$ $(f \in \mathcal{M}_{\mathrm{int}})$ *and* $g \mapsto g^\bullet$ $(g \in \mathcal{L}_{\mathrm{int}})$ *provide a*

one-to-one correspondence between $\mathcal{M}_{\rm int}$ *and* $\mathcal{L}_{\rm int}$*, and are the inverse of each other.*

Proof. We prove the claim (i) only, since (ii) can be shown similarly to (i), and (iii) is an immediate corollary to (i), (ii), and Theorem 4.1.

The equation (68) follows from the integral convexity of $f_{\mathbf{Z}}$ shown in Theorem 6.5. For any $p \in \mathrm{dom}\, f^\bullet$ Lemma 6.13 assures the existence of $x \in \mathrm{dom}\, f \cap \mathbf{Z}^V$ with $f^\bullet(p) = \langle p, x\rangle - f(x)$. We abbreviate $(f^\bullet)_{\mathbf{Z}}$ to $f^\bullet_{\mathbf{Z}}$. Theorem 6.12 and the equation (68) yield $f^\bullet_{\mathbf{Z}} \in \mathcal{L}[\mathbf{Z}]$. Therefore, it suffices to show $f^\bullet(p) = \overline{f^\bullet_{\mathbf{Z}}}(p)$ $(p \in \mathbf{R}^V)$.

We have $f^\bullet(p) = \overline{f^\bullet_{\mathbf{Z}}}(p)$ for integral vectors $p \in \mathbf{Z}^V$ and $f^\bullet(p) \le \overline{f^\bullet_{\mathbf{Z}}}(p)$ $(p \in \mathbf{R}^V)$. Hence, we may assume $f^\bullet(p) < +\infty$ since otherwise $f^\bullet(p) = \overline{f^\bullet_{\mathbf{Z}}}(p) = +\infty$ holds. Then, there exists some $x_0 \in \mathrm{dom}_{\mathbf{Z}}\, f_{\mathbf{Z}}$ such that $f^\bullet(p) = \langle p, x_0\rangle - f_{\mathbf{Z}}(x_0)$. Put

$$D = \{q \in \mathbf{R}^V \mid f^\bullet(q) = \langle q, x_0\rangle - f_{\mathbf{Z}}(x_0)\}.$$

Then, we have

$$D = \partial f_{\mathbf{Z}}(x_0) = \overline{\partial f_{\mathbf{Z}}(x_0) \cap \mathbf{Z}^V}, \quad \partial f_{\mathbf{Z}}(x_0) \cap \mathbf{Z}^V \in \mathcal{L}_0[\mathbf{Z}]$$

(see, e.g., [32, Theorem 4.7], [35, Theorem 2.41]), i.e., D is an integral polyhedron. Thus, there exist some integral vectors $\{q_i\}_{i=1}^k$ $(k \ge 1)$ in $\partial f_{\mathbf{Z}}(x_0) \cap \mathbf{Z}^V$ such that p can be represented by a convex combination of $\{q_i\}_{i=1}^k$. Since $f^\bullet$ is linear over D, $f^\bullet(q_i) = \overline{f^\bullet_{\mathbf{Z}}}(q_i)$ for $i = 1, \cdots, k$, and $f^\bullet(p) \le \overline{f^\bullet_{\mathbf{Z}}}(p)$, convexity of $f^\bullet$ implies $f^\bullet(p) = \overline{f^\bullet_{\mathbf{Z}}}(p)$. ▮

7. DUALITY

7.1. Duality Theorems

In this section, we discuss duality theorems for polyhedral M/L-convex functions.

We first state the Fenchel duality for convex/concave functions. For a convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and a concave function $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \ne \emptyset$, $\mathrm{dom}\, g \ne \emptyset$, we have

$$\begin{aligned} \inf_{x \in \mathbf{Z}^V} \{f(x) - g(x)\} &\ge \inf_{x \in \mathbf{R}^V} \{f(x) - g(x)\} \\ \ge \sup_{p \in \mathbf{R}^V} \{g^\circ(p) - f^\bullet(p)\} &\ge \sup_{p \in \mathbf{Z}^V} \{g^\circ(p) - f^\bullet(p)\}, \end{aligned} \tag{69}$$

where the *concave conjugate* $g^\circ : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$g^\circ(p) = \inf_{x \in \mathbf{R}^V} \{\langle p, x\rangle - g(x)\} \qquad (p \in \mathbf{R}^V).$$

The Fenchel duality (70) below shows that the second inequality in (69) is satisfied with equality under a certain assumption. This is just an application of the existing result in convex analysis to polyhedral convex/concave functions, and it stands independently of M-convexity/M-concavity or L-convexity/L-concavity of functions. On the other hand, the claim (ii) of Theorem 7.1 below shows that all inequalities in (69) are satisfied with equality if f and g are integral M-convex/M-concave. This integrality result is not obtained from the convexity/concavity alone, but from the combination of the convexity/concavity and the combinatorial properties of f and g.

THEOREM 7.1.
(i) *Let* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ *be polyhedral convex/concave functions with* $\mathrm{dom}\, f \neq \emptyset$, $\mathrm{dom}\, g \neq \emptyset$. *If either of* (a) $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ *and* (b) $\mathrm{dom}\, f^\bullet \cap \mathrm{dom}\, g^\circ \neq \emptyset$ *holds, then we have*

$$\inf_{x \in \mathbf{R}^V} \{f(x) - g(x)\} = \sup_{p \in \mathbf{R}^V} \{g^\circ(p) - f^\bullet(p)\}. \tag{70}$$

The supremum is attained at some $p_* \in \mathbf{R}^V$ *if* (a) *is satisfied, and the infimum is attained at some* $x_* \in \mathbf{R}^V$ *if* (b) *is satisfied.*
(ii) *If* $f, -g \in \mathcal{M}_{\mathrm{int}}$ *and either* (a) *or* (b) *is fulfilled, then we have*

$$\begin{aligned} \inf_{x \in \mathbf{Z}^V} \{f(x) - g(x)\} &= \inf_{x \in \mathbf{R}^V} \{f(x) - g(x)\} \\ = \sup_{p \in \mathbf{R}^V} \{g^\circ(p) - f^\bullet(p)\} &= \sup_{p \in \mathbf{Z}^V} \{g^\circ(p) - f^\bullet(p)\}. \end{aligned}$$

The supremum is attained at some integral $p_* \in \mathbf{Z}^V$ *if* (a) *is satisfied, and the infimum is attained at some integral* $x_* \in \mathbf{Z}^V$ *if* (b) *is satisfied.*

Proof. The claim (i) is a standard result; see [41, 44]. We derive (ii) from the corresponding result for discrete M-convex/concave functions established in [32].

We define the function $f_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ by (67), and $g_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ by

$$g_{\mathbf{Z}}(x) = \begin{cases} g(x) & (x \in \mathbf{Z}^V), \\ -\infty & (x \notin \mathbf{Z}^V). \end{cases}$$

Then, we have $f_{\mathbf{Z}}, -g_{\mathbf{Z}} \in \mathcal{M}[\mathbf{Z}]$. We also define $f^{\bullet}_{\mathbf{Z}} = (f^{\bullet})_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ and $g^{\circ}_{\mathbf{Z}} = (g^{\circ})_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ in the similar way, where it is noted that (cf. (68))

$$\begin{aligned} f^{\bullet}_{\mathbf{Z}}(p) &= \sup_{x \in \mathbf{Z}^V} \{\langle p, x\rangle - f_{\mathbf{Z}}(x)\} \qquad (p \in \mathbf{Z}^V), \\ g^{\circ}_{\mathbf{Z}}(p) &= \inf_{x \in \mathbf{Z}^V} \{\langle p, x\rangle - g_{\mathbf{Z}}(x)\} \qquad (p \in \mathbf{Z}^V). \end{aligned}$$

We have $\mathrm{dom}\, f = \overline{\mathrm{dom}_{\mathbf{Z}}\, f_{\mathbf{Z}}}$ and $\mathrm{dom}\, g = \overline{\mathrm{dom}_{\mathbf{Z}}\, g_{\mathbf{Z}}}$, and from Theorem 6.14 follows $\mathrm{dom}\, f^{\bullet} = \overline{\mathrm{dom}_{\mathbf{Z}}\, f^{\bullet}_{\mathbf{Z}}}$ and $\mathrm{dom}\, g^{\circ} = \overline{\mathrm{dom}_{\mathbf{Z}}\, g^{\circ}_{\mathbf{Z}}}$. These properties imply that the condition (a) is equivalent to $\mathrm{dom}_{\mathbf{Z}}\, f_{\mathbf{Z}} \cap \mathrm{dom}_{\mathbf{Z}}\, g_{\mathbf{Z}} \neq \emptyset$, and (b) is equivalent to $\mathrm{dom}_{\mathbf{Z}}\, f^{\bullet}_{\mathbf{Z}} \cap \mathrm{dom}_{\mathbf{Z}}\, g^{\circ}_{\mathbf{Z}} \neq \emptyset$. Thus, the first and the last term in (69) are equal by the Fenchel-type duality for discrete M-convex/M-concave functions [32, Theorem 5.2], and each inequality in (69) holds with equality. ▮

When f and g are polyhedral M-convex and M-concave, respectively, in (70), then $f^{\bullet}$ and g° are polyhedral L-convex and L-concave, respectively, by Theorem 5.1. Hence, Fenchel duality also reads that the minimization of the sum of polyhedral M-convex functions is equivalent to the minimization of the sum of their conjugate polyhedral L-convex functions. Note that the sum of polyhedral L-convex functions is also polyhedral L-convex, whereas the sum of polyhedral M-convex functions is not necessarily polyhedral M-convex (cf. Remark 3.5). As a corollary of Theorem 7.1, we obtain an optimality criterion of the minimization of the sum of two polyhedral convex functions.

COROLLARY 7.2. (i) *Let* $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$ *be a polyhedral convex function with* $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \neq \emptyset$. *Then, there exists* $p_* \in \mathbf{R}^V$ *such that*

$$\inf_{x \in \mathbf{R}^V} \{f_1(x) + f_2(x)\} = \inf_{x \in \mathbf{R}^V} f_1[-p_*](x) + \inf_{x \in \mathbf{R}^V} f_2[p_*](x). \tag{71}$$

(ii) *If* $f_1, f_2 \in \mathcal{M}_{\mathrm{int}}$ *and* $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \neq \emptyset$, *then there exists an integral* $p_* \in \mathbf{Z}^V$ *such that*

$$\begin{aligned} \inf_{x \in \mathbf{Z}^V} \{f_1(x) + f_2(x)\} &= \inf_{x \in \mathbf{R}^V} \{f_1(x) + f_2(x)\} \\ = \inf_{x \in \mathbf{R}^V} f_1[-p_*](x) + \inf_{x \in \mathbf{R}^V} f_2[p_*](x) &= \inf_{x \in \mathbf{Z}^V} f_1[-p_*](x) + \inf_{x \in \mathbf{Z}^V} f_2[p_*](x). \end{aligned}$$

REMARK 7.3. The conditions (a) and (b) in Theorem 7.1 cannot be removed, as shown in the following example.

Let $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{R}^2 \to \mathbf{R} \cup \{-\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} x_1 & (x_1 + x_2 = 1), \\ +\infty & (x_1 + x_2 \neq 1), \end{cases} \quad g(x_1, x_2) = \begin{cases} -x_1 & (x_1 + x_2 = -1), \\ -\infty & (x_1 + x_2 \neq -1). \end{cases}$$

Then, we have $f, -g \in \mathcal{M} \cap \mathcal{M}_{\text{int}}$, $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$, and

$$\inf_{(x_1,x_2)\in\mathbf{R}^2} \{f(x_1, x_2) - g(x_1, x_2)\} = +\infty.$$

On the other hand, it holds that

$$f^\bullet(p_1, p_2) = \begin{cases} p_2 & (p_1 - p_2 = 1), \\ +\infty & (p_1 - p_2 \neq 1), \end{cases} \quad g^\circ(p_1, p_2) = \begin{cases} -p_2 & (p_1 - p_2 = -1), \\ -\infty & (p_1 - p_2 \neq -1). \end{cases}$$

We have $f^\bullet, -g^\bullet \in \mathcal{L} \cap \mathcal{L}_{\text{int}}$, $\operatorname{dom} f^\bullet \cap \operatorname{dom} g^\circ = \emptyset$, and

$$\sup_{(p_1,p_2)\in\mathbf{R}^2} \{g^\circ(p_1, p_2) - f^\bullet(p_1, p_2)\} = -\infty.$$

Neither (a) nor (b) is fulfilled, and the equation (70) does not hold. ▮

We state separation theorems for polyhedral convex/concave functions. Whereas the claim (i) of Theorems 7.4 below is just the standard separation theorem in convex analysis, (ii) and (iii) are based on the combinatorial properties of M/L-convexity and their essence lies in the discrete separation theorems shown in [29, 30, 32]. See also [18] for the proof of (ii) and (iii).

THEOREM 7.4. *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ be functions with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Suppose $f(x) \geq g(x)$ ($\forall x \in \mathbf{R}^V$) holds.*
(i) *If f and g are polyhedral convex and concave, respectively, then there exist $p_* \in \mathbf{R}^V$ and $\alpha_* \in \mathbf{R}$ such that*

$$f(x) \geq \langle p_*, x\rangle + \alpha_* \geq g(x) \qquad (x \in \mathbf{R}^V). \tag{72}$$

(ii) *If $f, -g \in \mathcal{M}_{\text{int}}$, then we can take integral $p_* \in \mathbf{Z}^V$ and $\alpha_* \in \mathbf{Z}$ in (72).*
(iii) *If $f, -g \in \mathcal{L}_{\text{int}}$, then we can take integral $p_* \in \mathbf{Z}^V$ and $\alpha_* \in \mathbf{Z}$ in (72).*

7.2. Submodular Flow Problem with M-convex Cost Function

We consider a generalization of the ordinary submodular flow problem and state a duality theorem for the problem. We also provide two optimality criteria, one of which is by negative cycles and the other by potentials.

The generalization here is a polyhedral extension of the framework proposed in [29]. See, e.g., [3, 10, 12, 17, 40] for the ordinary submodular flow problem.

Let $G = (V, A)$ be a directed graph. A *flow* is a vector $\xi \in \mathbf{R}^A$. Suppose that we are given a flow cost function $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ for each $a \in A$ and a boundary cost function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$. Then, the minimum cost flow problem with a boundary cost function is formulated as follows:

$$\text{(MFBC) Minimize} \quad F(\xi) = f(\partial\xi) + \sum_{a \in A} f_a(\xi(a)) \quad \text{subject to} \quad \xi \in \mathbf{R}^A,$$

where $\partial\xi \in \mathbf{R}^V$ is the boundary of ξ defined by (1). The problem (MFBC) coincides with the ordinary (linear) submodular flow problem if each f_a is a linear function in a nonempty closed interval and f is the indicator function of some M-convex polyhedron, i.e., f is a polyhedral M-convex function with $f : \mathbf{R}^V \to \{0, +\infty\}$. We call a flow ξ *feasible* if $F(\xi) < +\infty$.

We then consider the dual of the problem (MFBC). A *potential* is a vector $p \in \mathbf{R}^V$. Suppose that we are given a tension cost function $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ for each $a \in A$ and a potential cost function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$. Then, the minimum cost tension problem with a potential cost function is formulated as follows:

$$\text{(MTPC) Minimize} \quad G(p) = g(p) + \sum_{a \in A} g_a(-\delta p(a)) \quad \text{subject to} \quad p \in \mathbf{R}^V,$$

where $\delta p \in \mathbf{R}^A$ is the coboundary of p defined by (3). The problem (MTPC) coincides with the ordinary minimum (convex-)cost tension problem if $g(p) = 0$ for any $p \in \mathbf{R}^V$. We call a potential p *feasible* if $G(p) < +\infty$.

Theorem 7.1 (i) yields the duality between (MFBC) and (MTPC). Note that this is independent of M/L-convexity.

THEOREM 7.5. *Let f and g be polyhedral convex functions such that* $\operatorname{dom} f \neq \emptyset$, $\operatorname{dom} g \neq \emptyset$, *and* $g = f^\bullet$, *and for each $a \in A$ let $f_a, g_a \in \mathcal{C}^1$ satisfy $g_a = f_a^\bullet$. Suppose that at least one of two problems* (MFBC) *and* (MTPC) *has a feasible solution. Then,*

$$\inf_{\xi \in \mathbf{R}^A} F(\xi) = - \inf_{p \in \mathbf{R}^V} G(p).$$

If (MFBC) *is feasible, then $G(p_*) = \inf G(p)$ for some $p_* \in \mathbf{R}^V$, and if* (MTPC) *is feasible, then $F(\xi_*) = \inf F(\xi)$ for some $\xi_* \in \mathbf{R}^A$.*

Proof. Define functions $f_1, f_2 : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_1(x) = \inf_{\xi \in \mathbf{R}^A} \{\sum_{a \in A} f_a(\xi(a)) \mid \partial\xi = -x\}, \quad f_2(x) = f(-x) \ (x \in \mathbf{R}^V). \quad (73)$$

Similarly, we define functions $g_1, g_2 : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$g_1(p) = \sum_{a \in A} g_a(-\delta p(a)), \quad g_2(p) = g(p) \qquad (p \in \mathbf{R}^V).$$

Note that f_1 and g_1 are special cases of functions f and g in Example 2.4, where $T = V$. Moreover, if (MFBC) is feasible, then $f_1(x_0)$ is finite for some $x_0 \in \mathbf{R}^V$, and if (MTPC) is feasible, then $g_1(p_0)$ is finite for some $p_0 \in \mathbf{R}^V$. Therefore, f_1 and g_1 are polyhedral convex functions such that $f_1 > -\infty$, $g_1 > -\infty$, $\mathrm{dom}\, f_1 \neq \emptyset$, $\mathrm{dom}\, g_1 \neq \emptyset$, and $g_1 = f_1^{\bullet}$. Moreover, we have $(-f_2)^{\circ}(p) = -g_2(p)$ $(\forall p \in \mathbf{R}^V)$. Thus, Theorem 7.1 (i) yields the following equation:

$$\begin{aligned} \inf_{\xi \in \mathbf{R}^A} F(\xi) &= \inf_{x \in \mathbf{R}^V} \{f_1(x) + f_2(x)\} \\ &= -\inf_{p \in \mathbf{R}^V} \{g_2(p) + g_1(p)\} \ = \ -\inf_{p \in \mathbf{R}^V} G(p). \end{aligned}$$

∎

To state the first optimality criterion, we define the *auxiliary network* $(G_\xi = (V, A_\xi), w_\xi)$ associated with a feasible flow $\xi \in \mathbf{R}^A$. The underlying graph G_ξ has the vertex set V and the arc set A_ξ consisting of three disjoint parts: $A_\xi = F_\xi \cup B_\xi \cup J_\xi$, where

$$\begin{aligned} F_\xi &= \{a \mid a \in A,\ f'_a(\xi(a); +1) < +\infty\}, \\ B_\xi &= \{\bar{a} \mid a \in A,\ f'_a(\xi(a); -1) < +\infty\} \qquad (\bar{a} : \text{ reorientation of } a), \\ J_\xi &= \{(u, v) \mid u, v \in V,\ u \neq v,\ f'(\partial\xi; v, u) < +\infty\}. \end{aligned}$$

The *weight* function $w_\xi : A_\xi \to \mathbf{R}$ is defined by

$$w_\xi(a) = \begin{cases} f'_a(\xi(a); +1) & (a \in F_\xi), \\ f'_{\bar{a}}(\xi(\bar{a}); -1) & (a \in B_\xi), \\ f'(\partial\xi; v, u) & (a \in J_\xi). \end{cases}$$

For any cycle C $(\subseteq A_\xi)$ in G_ξ, we define the *weight* of C by $w_\xi(C) = \sum\{w_\xi(a) \mid a \in C\}$, and call C a *negative cycle* if $w_\xi(C) < 0$.

THEOREM 7.6. *Let f be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$, *and $\xi \in \mathbf{R}^A$ be a feasible flow.*

(i) *There is no negative cycle in* (G_ξ, w_ξ) *if* ξ *is optimal for* (MFBC).
(ii) *The converse is also true if* f *is M-convex. That is, for* $f \in \mathcal{M}$:
ξ *is optimal for* (MFBC) $\iff$ *there is no negative cycle in* (G_ξ, w_ξ).

Proof. (i): Suppose that there exists a negative cycle C ($\subseteq A_\xi$) in G_ξ. We show that $F(\xi_*) < F(\xi)$ holds for some flow $\xi_* \in \mathbf{R}^A$. Put

$$C_F = C \cap F_\xi, \quad C_B = C \cap B_\xi, \quad C_J = C \cap J_\xi, \quad y = \sum_{(u,v)\in C_J} (\chi_v - \chi_u).$$

Then, there exists a sufficiently small $\alpha > 0$ such that

$$f(\partial\xi + \alpha y) - f(\partial\xi) \;=\; \alpha f'(\partial\xi; y) \;\le\; \alpha \sum_{(u,v)\in C_J} w_\xi(u,v), \qquad (74)$$

$$f_a(\xi(a) + \alpha) - f_a(\xi(a)) \;=\; \alpha w_\xi(a) \qquad (a \in C_F), \qquad (75)$$

$$f_{\bar a}(\xi(\bar a) - \alpha) - f_{\bar a}(\xi(\bar a)) \;=\; \alpha w_\xi(\bar a) \qquad (a \in C_B), \qquad (76)$$

where the inequality in (74) is from Theorems 4.2 and 4.4. We define a flow $\xi_* \in \mathbf{R}^A$ by

$$\xi_*(a) = \begin{cases} \xi(a) + \alpha & (a \in C_F), \\ \xi(\bar a) - \alpha & (a \in C_B), \\ \xi(a) & (\text{otherwise}). \end{cases}$$

Combining the inequalities (74), (75), and (76), we obtain

$$\begin{aligned} F(\xi_*) - F(\xi) &= \{f(\partial\xi + \alpha y) - f(\partial\xi)\} + \sum_{a\in C_F} \{f_a(\xi(a) + \alpha) - f_a(\xi(a))\} \\ &\qquad + \sum_{a\in C_B} \{f_{\bar a}(\xi(\bar a) - \alpha) - f_{\bar a}(\xi(\bar a))\} \\ &\le \alpha w_\xi(C) < 0. \end{aligned}$$

(ii): Suppose that there exists a flow $\xi_* \in \mathbf{R}^V$ with $F(\xi_*) < F(\xi)$. To prove the existence of a negative cycle in (G_ξ, w_ξ), it suffices to show the existence of a "circulation" in (G_ξ, w_ξ) with negative cost w.r.t. w_ξ, i.e., the existence of $\zeta : A_\xi \to \mathbf{R}$ satisfying $\zeta \ge 0$, $\sum_{a\in A_\xi} w_\xi(a)\zeta(a) < 0$ and

$$\sum\{\zeta(a) \mid a(\in A_\xi) \text{ leaves } v\} = \sum\{\zeta(a) \mid a(\in A_\xi) \text{ enters } v\} \ (v \in V). \ (77)$$

We have

$$\begin{aligned} f_a(\xi_*(a)) - f_a(\xi(a)) \ &\geq \ \{\xi_*(a) - \xi(a)\} f_a'(\xi(a); +1) \\ & \qquad\qquad (\forall a \in \mathrm{supp}^+(\xi_* - \xi)), \qquad (78) \\ f_a(\xi_*(a)) - f_a(\xi(a)) \ &\geq \ \{\xi(a) - \xi_*(a)\} f_a'(\xi(a); -1) \\ & \qquad\qquad (\forall a \in \mathrm{supp}^-(\xi_* - \xi)), \qquad (79) \\ f(\partial\xi_*) - f(\partial\xi) \ &\geq \ f'(\partial\xi; \partial\xi_* - \partial\xi). \qquad (80) \end{aligned}$$

By Theorem 4.15 (ii), there exists some $\lambda = (\lambda_{uv} \mid (u,v) \in J_\xi)$ such that

$$\sum_{(u,v)\in J_\xi} \lambda_{uv}(\chi_v - \chi_u) = \partial\xi_* - \partial\xi, \qquad \lambda_{uv} \geq 0 \quad ((u,v) \in J_\xi),$$

$$\sum_{(u,v)\in J_\xi} \lambda_{uv} f'(\partial\xi; v, u) = f'(\partial\xi; \partial\xi_* - \partial\xi). \qquad (81)$$

We now define a function $\zeta : A_\xi \to \mathbf{R}$ as follows:

$$\zeta(a) = \begin{cases} \max\{\xi_*(a) - \xi(a), 0\} & (a \in F_\xi), \\ \max\{\xi(\bar{a}) - \xi_*(\bar{a}), 0\} & (a \in B_\xi), \\ \lambda_{uv} & (a = (u,v) \in J_\xi). \end{cases}$$

It is easy to see that $\zeta \geq 0$ and ζ satisfies (77). Moreover, (78), (79), and (81) imply

$$\begin{aligned} \sum_{a\in A_\xi} w_\xi(a)\zeta(a) \ &= \sum_{a\in\mathrm{supp}^+(\xi_*-\xi)} \{\xi_*(a) - \xi(a)\} f_a'(\xi(a); +1) \\ &\quad + \sum_{a\in\mathrm{supp}^-(\xi_*-\xi)} \{\xi(a) - \xi_*(a)\} f_a'(\xi(a); -1) + \sum_{(u,v)\in J_\xi} \lambda_{uv} f'(\partial\xi; v, u) \\ &\leq \ F(\xi_*) - F(\xi) \ < \ 0. \end{aligned}$$

Hence, ζ is a circulation in (G_ξ, w_ξ) with negative cost. ▮

REMARK 7.7. The claim (ii) of Theorem 7.6 does not hold in general when $f \notin \mathcal{M}$.

Let $G = (V, A)$ be a directed graph with $V = \{u, v, w\}$ and $A = \{(u,w), (v,w)\}$, $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$ a family of flow cost functions such that

$$\mathrm{dom}\, f_a = [0, 1], \qquad f_a(\alpha) = 0 \quad (\alpha \in \mathrm{dom}\, f_a),$$

and $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ a boundary cost function such that

$$f(x) = \begin{cases} -\min\{x(u), x(v)\} & (x(u) \geq 0,\ x(v) \geq 0,\ x(V) = 0), \\ +\infty & (\text{otherwise}). \end{cases}$$

It is easy to see that f is polyhedral convex, but not polyhedral M-convex.

For a feasible flow $\xi = (0,0) \in \mathbf{R}^A$, the auxiliary network $(G_\xi = (V, A_\xi), w_\xi)$ is such that

$$A_\xi = \{(u,w),(v,w),(w,u),(w,v)\}, \qquad w_\xi(a) = 0 \quad (a \in A_\xi).$$

There is no negative cycle in (G_ξ, w_ξ), but ξ is not optimal; a feasible flow $\xi_* \in \mathbf{R}^A$ is optimal if and only if either $\xi_*(u,w) = 1$ or $\xi_*(v,w) = 1$ holds. ▮

We next state an optimality criterion by potentials. Recall the definition of the coboundary δp of a potential $p \in \mathbf{R}^V$ in (3).

THEOREM 7.8. *Let $\xi_* \in \mathbf{R}^A$ be a feasible flow.*
(i) *If the condition*

$$\partial\xi_* \in \arg\min f[-p_*], \qquad \xi_*(a) \in \arg\min f_a[-\eta(a)] \quad (\forall a \in A) \quad (82)$$

holds for some $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$, then ξ_* is optimal for (MFBC).
(ii) *Suppose that f is polyhedral convex with* $\mathrm{dom}\, f \neq \emptyset$, *and $f_a \in \mathcal{C}^1$ $(a \in A)$. Then, ξ_* is optimal for* (MFBC) *if and only if (82) holds for some $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$.*
(iii) *If $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$ satisfy (82), then we have*

$$p_*(v) - p_*(u) \leq w_\xi(u,v) \qquad (\forall (u,v) \in A_{\xi_*}). \quad (83)$$

(iv) *Suppose $f \in \mathcal{M}$ and $f_a \in \mathcal{C}^1$ $(a \in A)$. Then, ξ_* is optimal for* (MFBC) *if and only if (83) holds for some $p_* \in \mathbf{R}^V$.*

Proof. Note that

$$F(\xi) = f[-p_*](\partial\xi) + \sum_{a \in A} f_a[-\eta(a)](\xi(a))$$

for any $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$. Hence, the claim (i) follows immediately. The claim (iii) is also immediate from the definition of w_ξ.

(ii): Suppose ξ_* is optimal for (MFBC). Define $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ $(i = 1, 2)$ by

$$f_1(x) = f(x), \quad f_2(x) = \inf_{\xi \in \mathbf{R}^A} \{\sum_{a \in A} f_a(\xi(a)) \mid \partial\xi = x\} \quad (x \in \mathbf{R}^V).$$

Then, Corollary 7.2 implies the existence of $p_* \in \mathbf{R}^V$ with (71). Note that

$$\begin{aligned} \inf_{x \in \mathbf{R}^V} f_2[p_*](x) &= \inf_{\xi \in \mathbf{R}^A} \{\langle p_*, \partial\xi \rangle_V + \sum_{a \in A} f_a(\xi(a))\} \\ &= \sum_{a \in A} \inf_{\alpha \in \mathbf{R}} \{\delta p_*(a)\alpha + f_a(\alpha)\}, \end{aligned}$$

where the second equality is by $\langle p_*, \partial\xi \rangle_V = \langle \delta p_*, \xi \rangle_A$. Hence, we have (82).

(iv): By Theorem 4.12 for $f \in \mathcal{M}$ and the convexity of f_a $(a \in A)$, the condition (82) is equivalent to (83). Hence, the claim follows from (ii). ▮

Remark 7.9. The following claim is easy to show: for a feasible flow $\xi \in \mathbf{R}^A$, if there exists $p_* \in \mathbf{R}^V$ with (83), then there is no negative cycle in the auxiliary network (G_ξ, w_ξ). Indeed, for any cycle C in (G_ξ, w_ξ)

$$w_\xi(C) = \sum_{(u,v) \in C} w_\xi(u, v) \geq \sum_{(u,v) \in C} \{p_*(v) - p_*(u)\} = 0$$

follows from (83). The converse is also a well-known fact in graph theory. ▮

Summarizing the above results for $f \in \mathcal{M}$, we obtain the following statement.

Corollary 7.10. *Suppose $f \in \mathcal{M}$. For any feasible flow $\xi \in \mathbf{R}^A$, the following three conditions are equivalent:*

(OPT) *ξ is optimal for (MFBC).*

(NNC) *There is no negative cycle in the auxiliary network (G_ξ, w_ξ).*

(POT) *There exists a potential $p \in \mathbf{R}^V$ with (83).*

7.3. Network Induction

As shown in Section 4 (Theorems 4.18 and 4.35 in particular), there are various operations for polyhedral M/L-convex functions. In this section, we explain a more powerful operation called *network induction*. Network induction is a transformation by using networks, and includes many operations such as translation, restriction, projection, etc., as its special cases.

We show that polyhedral M/L-convex functions can be transformed into polyhedral M/L-convex functions by network induction.

Let $G = (V, A; S, T)$ be a directed graph with vertex set V, arc set A, and two disjoint vertex subsets S and T called entrance set and exit set, respectively. For $f : \mathbf{R}^S \to \mathbf{R} \cup \{+\infty\}$ and $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, define $\tilde{f} : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{f}(y) \; &= \; \inf_{\xi, x}\{f(x) + \sum_{a \in A} f_a(\xi(a)) \mid \partial\xi = (x, -y, \mathbf{0}), \; \xi \in \mathbf{R}^A, \\ &\qquad (x, -y, \mathbf{0}) \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)}\} \quad (y \in \mathbf{R}^T). \quad (84)\end{aligned}$$

For $g : \mathbf{R}^S \to \mathbf{R} \cup \{+\infty\}$ and $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, define $\tilde{g} : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{g}(q) = \inf_{\eta, p, r}\{g(p) + \sum_{a \in A} g_a(\eta(a)) \mid \eta = -\delta(p, q, r) \in \mathbf{R}^A, \\ (p, q, r) \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)}\} \quad (q \in \mathbf{R}^T).\end{aligned}$$

THEOREM 7.11. *Let f, g be polyhedral convex functions such that* $\mathrm{dom}\, f \neq \emptyset$, $\mathrm{dom}\, g \neq \emptyset$, *and $g = f^{\bullet}$, and $f_a, g_a \in \mathcal{C}^1$ with $g_a = f_a^{\bullet}$ $(a \in A)$. Suppose that at least one of the following three conditions holds:*

(a) $\tilde{f}(y_0)$ *is finite for some* $y_0 \in \mathbf{R}^T$,

(b) $\tilde{g}(q_0)$ *is finite for some* $q_0 \in \mathbf{R}^T$,

(c) $\tilde{f}(y_0) < +\infty$ *and* $\tilde{g}(q_0) < +\infty$ *for some* $y_0 \in \mathbf{R}^T$ *and* $q_0 \in \mathbf{R}^T$.

Then, the following statements hold:

(i) *both $\tilde{f}$ and $\tilde{g}$ are polyhedral convex functions such that* $\mathrm{dom}\, \tilde{f} \neq \emptyset$, $\mathrm{dom}\, \tilde{g} \neq \emptyset$, $\tilde{f} > -\infty$, $\tilde{g} > -\infty$, *and* $\tilde{g} = (\tilde{f})^{\bullet}$.

(ii) *if f is polyhedral M-convex, then $\tilde{f}$ is also polyhedral M-convex.*

(iii) *if g is polyhedral L-convex, then $\tilde{g}$ is also polyhedral L-convex.*

Proof. (i): We first prove two claims.

Claim 1 For $y \in \mathbf{R}^T$, we have $\tilde{f}(y) = (\tilde{g})^{\bullet}(y)$ if either $\tilde{f}(y) < +\infty$ or $\tilde{g}(q') < +\infty$ for some $q' \in \mathbf{R}^T$.

Define a function $\hat{f} : \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)} \to \mathbf{R} \cup \{+\infty\}$ by

$$\hat{f}(x', y', z') = \begin{cases} f(x') & (y' = -y, \; z' = \mathbf{0}), \\ +\infty & (\text{otherwise}). \end{cases}$$

Then, its conjugate $(\hat{f})^{\bullet} : \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)} \to \mathbf{R} \cup \{+\infty\}$ is given by

$$(\hat{f})^{\bullet}(p', q', r') = g(p') - \langle q', y \rangle_T \quad ((p', q', r') \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)}).$$

By Theorem 7.5, we have

$$\begin{aligned}\tilde{f}(y) &= \inf\{\hat{f}(\partial\xi) + \sum_{a\in A} f_a(\xi(a)) \mid \xi \in \mathbf{R}^V\} \\ &= -\inf\{(\hat{f})^\bullet(p',q',r') + \sum_{a\in A} g_a(\eta(a)) \mid \eta = -\delta(p',q',r') \in \mathbf{R}^A, \\ &\qquad\qquad (p',q',r') \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V\setminus(S\cup T)}\} \\ &= -\inf\{-\langle q', y\rangle_T + \tilde{g}(q') \mid q' \in \mathbf{R}^T\} \;=\; (\tilde{g})^\bullet(y).\end{aligned}$$

[End of Proof for Claim 1]

Claim 2 For $q \in \mathbf{R}^T$, we have $\tilde{g}(q) = (\tilde{f})^\bullet(q)$ if either $\tilde{g}(q) < +\infty$ or $\tilde{f}(y') < +\infty$ for some $y' \in \mathbf{R}^T$.

Define a function $\hat{g} : \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V\setminus(S\cup T)} \to \mathbf{R} \cup \{+\infty\}$ by

$$\hat{g}(p',q',r') = \begin{cases} g(p') & (q' = q), \\ +\infty & (\text{otherwise}). \end{cases}$$

Then, its conjugate $(\hat{g})^\bullet : \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V\setminus(S\cup T)} \to \mathbf{R} \cup \{+\infty\}$ is given by

$$(\hat{g})^\bullet(x',y',z') = \begin{cases} f(x') + \langle q, y'\rangle & (z' = \mathbf{0}), \\ +\infty & (\text{otherwise}). \end{cases}$$

By Theorem 7.5, we have

$$\begin{aligned}\tilde{g}(q) &= \inf\{\hat{g}(p',q',r') + \sum_{a\in A} g_a(\eta(a)) \mid \eta = -\delta(p',q',r') \in \mathbf{R}^A, \\ &\qquad\qquad (p',q',r') \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V\setminus(S\cup T)}\} \\ &= -\inf\{(\hat{g})^\bullet(x',y',z') + \sum_{a\in A} f_a(\xi(a)) \mid \partial\xi = (x',y',z'),\ \xi \in \mathbf{R}^A, \\ &\qquad\qquad (x',y',z') \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V\setminus(S\cup T)}\} \\ &= -\inf\{\langle q, y'\rangle_T + \tilde{f}(-y') \mid y' \in \mathbf{R}^T\} \;=\; (\tilde{f})^\bullet(q).\end{aligned}$$

[End of Proof for Claim 2]

We see from Claim 1 (resp. Claim 2) that the conditions (a) and (c) (resp. (b) and (c)) are equivalent. Hence, the statement (i) follows.

(ii): Suppose that g is polyhedral L-convex. Define $r_0 = \{g(p + \lambda\mathbf{1}) - g(p)\}/\lambda$, which is independent of $p \in \mathbf{R}^S$ and $\lambda \in \mathbf{R}$. Since

$$\delta(p + \lambda\mathbf{1}, q + \lambda\mathbf{1}, r + \lambda\mathbf{1}) = \delta(p,q,r) \quad (\forall\lambda \in \mathbf{R}),$$

we have

$$\begin{aligned}\tilde{g}(q+\lambda\mathbf{1}) &= \inf_{\eta,p',r'}\{g(p') + \sum_{a\in A} g_a(\eta(a)) \mid \eta = -\delta(p', q+\lambda\mathbf{1}, r')\} \\ &= \inf_{\eta,p,r}\{g(p+\lambda\mathbf{1}) + \sum_{a\in A} g_a(\eta(a)) \mid \eta = -\delta(p,q,r)\} \\ &= \tilde{g}(q) + \lambda r_0.\end{aligned}$$

For $q_1, q_2 \in \mathrm{dom}\,\tilde{g}$, we show $\tilde{g}(q_1) + \tilde{g}(q_2) \geq \tilde{g}(q_1 \wedge q_2) + \tilde{g}(q_1 \vee q_2)$. For $i = 1, 2$, let $(p_i, q_i, r_i) \in \mathbf{R}^V$ satisfy $\tilde{g}(q_i) = g(p_i) + \sum_{a\in A} g_a(\eta_i(a))$, where $\eta_i = -\delta(p_i, q_i, r_i)$. Putting

$$\eta_\wedge = -\delta(p_1 \wedge p_2, q_1 \wedge q_2, r_1 \wedge r_2), \quad \eta_\vee = -\delta(p_1 \vee p_2, q_1 \vee q_2, r_1 \vee r_2),$$

we obtain

$$g_a(\eta_1(a)) + g_a(\eta_2(a)) \geq g_a(\eta_\wedge(a)) + g_a(\eta_\vee(a)) \quad (a \in A)$$

by the convexity of g_a, while

$$g(p_1) + g(p_2) \geq g(p_1 \wedge p_2) + g(p_1 \vee p_2)$$

holds by the submodularity of g. Hence follows

$$\begin{aligned}\tilde{g}(q_1) + \tilde{g}(q_2) &\geq g(p_1 \wedge p_2) + g(p_1 \vee p_2) + \sum_{a\in A}\{g_a(\eta_\wedge(a)) + g_a(\eta_\vee(a))\} \\ &\geq \tilde{g}(q_1 \wedge q_2) + \tilde{g}(q_1 \vee q_2).\end{aligned}$$

(iii): Immediate from (i), (iii), and Theorem 5.1. ▮

Proof of Theorem 4.18 (7) Consider the bipartite graph $G = (V \cup \{v_0\} \cup U', A; S, T)$ with $S = V$, $T = U' \cup \{v_0\}$, and $A = \{(v, v_0) \mid v \in V \setminus U\} \cup \{(u, u') \mid u \in U\}$, where $u' \in U'$ is the copy of $u \in U$. Then, we have $\hat{f}(y_0, y) = \tilde{f}(-y_0, -y)$ for $(y_0, y) \in \mathbf{R} \times \mathbf{R}^U$, where $\hat{f}$ is defined by (38) and $\tilde{f}$ is the induced function defined by (84). Thus, $\hat{f}$ is polyhedral M-convex by Theorem 7.11 (ii). ▮

REMARK 7.12. The convolution of two functions can be represented as a special case of network induction, and vice versa.

We first show how to represent the convolution of two functions as a network induction. Let $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$. Let V_1 and V_2 be disjoint copies of V and consider a bipartite graph $G = (V_1 \cup V_2 \cup V, A; S, T)$ with $S = V_1 \cup V_2$, $T = V$, and $A = \{(v_1, v) \mid v \in V\} \cup \{(v_2, v) \mid v \in V\}$, where $v_i \in V_i$ is the copy of $v \in V$ $(i = 1, 2)$. Define $f : \mathbf{R}^{V_1 \cup V_2} \to \mathbf{R} \cup \{+\infty\}$ and $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$ by

$$f(x_1, x_2) = f_1(x_1) + f_2(x_2) \quad (x_i \in \mathbf{R}^{V_i}), \qquad f_a(\alpha) = 0 \quad (\alpha \in \mathbf{R}).$$

Then, $f_1 \Box f_2 = \tilde{f}$, where $\tilde{f}$ is the induced function defined by (84).

We next show that the function $\tilde{f}$ obtained by the network induction (84) can be represented as the convolution of two functions. Suppose we are given a directed graph $G = (V, A; S, T)$ with an entrance set S and an exit set T, and functions $f : \mathbf{R}^S \to \mathbf{R} \cup \{+\infty\}$, $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$. We define $\hat{f}_i : \mathbf{R}^{S \cup T} \to \mathbf{R} \cup \{\pm\infty\}$ $(i = 1, 2)$ as

$$\begin{aligned}
\hat{f}_1(x, y) &= \begin{cases} f(x) & (y = \mathbf{0}_T), \\ +\infty & (\text{otherwise}) \end{cases} \qquad (x \in \mathbf{R}^S,\ y \in \mathbf{R}^T), \\
\hat{f}_2(x, y) &= \inf \left\{ \sum_{a \in A} f_a(\xi(a)) \;\middle|\; \begin{array}{l} \xi \in \mathbf{R}^A, \\ \partial\xi = (-x, -y, \mathbf{0}) \end{array} \right\} \quad (x \in \mathbf{R}^S,\ y \in \mathbf{R}^T).
\end{aligned}$$

Then, the convolution $\hat{f}_1 \Box \hat{f}_2 : \mathbf{R}^{S \cup T} \to \mathbf{R} \cup \{\pm\infty\}$ is written as

$$\begin{aligned}
(\hat{f}_1 \Box \hat{f}_2)(x, y) &= \inf \left\{ \hat{f}_1(x_1, y_1) + \hat{f}_2(x_2, y_2) \;\middle|\; \begin{array}{l} x_1 + x_2 = x,\ x_i \in \mathbf{R}^S, \\ y_1 + y_2 = y,\ \ y_i \in \mathbf{R}^T \end{array} \right\} \\
&= \inf\{\hat{f}_1(x_1, \mathbf{0}_T) + \hat{f}_2(x_2, y) \mid x_1 + x_2 = x,\ x_i \in \mathbf{R}^S\} \\
&= \inf\{f(x') + \hat{f}_2(x - x', y) \mid x' \in \mathbf{R}^S\} \\
&= \inf \left\{ f(x') + \sum_{a \in A} f_a(\xi(a)) \;\middle|\; \begin{array}{l} \xi \in \mathbf{R}^A,\ x' \in \mathbf{R}^S, \\ \partial\xi = (x' - x, -y, \mathbf{0}) \end{array} \right\}
\end{aligned}$$

for $x \in \mathbf{R}^S$ and $y \in \mathbf{R}^T$. Thus, $(\hat{f}_1 \Box \hat{f}_2)(\mathbf{0}_S, y) = \tilde{f}(y)$ holds for $y \in \mathbf{R}^T$. ∎

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