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Conjugacy Relationship Between M-convex and L-convex Functions in Continuous Variables ⋆

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Abstract. By extracting combinatorial structures in well-solved nonlinear combinatorial optimization problems, Murota (1996,1998) introduced the concepts of M-convexity and L-convexity to functions defined over the integer lattice. Recently, Murota–Shioura (2000, 2001) extended these concepts to polyhedral convex functions and quadratic functions in continuous variables. In this paper, we consider a further extension to more general convex functions defined over the real space, and provide a proof for the conjugacy relationship between general M-convex and L-convex functions.

Key words. combinatorial optimization – matroid – base polyhedron – convex function – convex analysis

1. Introduction

Combinatorial optimization problems with nonlinear objective functions have been dealt with more often than before due to theoretical interest and the needs of practical applications. Extensive studies have been done for revealing the essence of the well-solvability in nonlinear combinatorial optimization problems [2,4,6–8,12,13,21]. By extracting combinatorial structures in well-solved nonlinear combinatorial optimization problems, Murota [9,10] introduced the concepts of M-convexity and L-convexity for functions defined over the integer lattice; subsequently, their variants called M$^\natural$-convexity and L$^\natural$-convexity were introduced by Murota–Shioura [14] and by Fujishige–Murota [5], respectively. Applications of M-/L-convexity can be found in mathematical economics with indivisible commodities [3,18,19], system analysis by mixed polynomial matrices [11], etc. Recently, Murota–Shioura [15,16] extended these concepts to polyhedral convex functions and quadratic functions defined over the real space. In this paper, we consider a further extension to more general convex functions defined over the real space.

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The concepts of M-convexity and L-convexity are defined for polyhedral convex functions and quadratic functions as follows. Let n be a positive integer, and put $N = \{1, 2, \dots, n\}$. A polyhedral convex function (or quadratic function) $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if $\mathrm{dom}\, f$ is nonempty and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$f(x)+f(y) \geq f(x-\alpha(\chi_i-\chi_j))+f(y+\alpha(\chi_i-\chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]), \quad (1.1)$$

where

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\},$$
$$\mathrm{supp}^+(x-y) = \{i \in N \mid x(i) > y(i)\}, \quad \mathrm{supp}^-(x-y) = \{i \in N \mid x(i) < y(i)\},$$

$x(i)$ is the i-th component of a vector $x \in \mathbf{R}^n$ for $i \in N$, and $\chi_i \in \{0,1\}^n$ is the i-th unit vector for $i \in N$. On the other hand, a polyhedral convex function (or quadratic function) $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p+\lambda\mathbf{1}) = g(p)+\lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$,

where $p \wedge q, p \vee q \in \mathbf{R}^n$ are defined by

$$(p \wedge q)(i) = \min\{p(i), q(i)\}, \quad (p \vee q)(i) = \max\{p(i), q(i)\} \quad (i \in N),$$

and $\mathbf{1} \in \mathbf{R}^n$ is the vector with all components equal to one.

To fully cover the well-solved nonlinear combinatorial optimization problems, it is desirable to further extend these concepts to more general convex functions defined over the real space on the basis of (M-EXC), and (LF1) and (LF2), respectively. It can be easily imagined that the previous results of M-/L-convexity for polyhedral convex functions and quadratic functions naturally extend to more general M-/L-convex functions. In particular, it is natural to imagine that the conjugacy relationship holds for general M-convex and L-convex functions over the real space, as in the cases of functions over the integer lattice [10, Th. 4.24], polyhedral convex functions [15, Th. 5.1], and quadratic functions [16, Th. 4.1]. However, the proof cannot be extended so directly to general M-/L-convex functions, but some technical difficulties such as topological issues arise. By taking such technical difficulties into consideration, we define M-convex and L-convex functions over the real space as convex functions satisfying (M-EXC), and (LF1) and (LF2), respectively. The primary contribution of this paper is to provide a rigorous proof of the following conjugacy relationship between general M-convex and L-convex functions over the real space.

Theorem 1.1. *For $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, define its conjugate function $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ by*

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n), \qquad (1.2)$$

where $\langle p, x\rangle = \sum_{i=1}^{n} p(i)x(i)$*.*
(i) *If* f *is a closed proper M-convex function, then* $f^{\bullet}$ *is a closed proper L-convex function with* $(f^{\bullet})^{\bullet} = f$*.*
(ii) *If* g *is a closed proper L-convex function, then* $g^{\bullet}$ *is a closed proper M-convex function with* $(g^{\bullet})^{\bullet} = g$*.*
(iii) *The mappings* $f \mapsto f^{\bullet}$ *(*f *: M-convex) and* $g \mapsto g^{\bullet}$ *(*g *: L-convex) provide a one-to-one correspondence between the classes of closed proper M-convex and L-convex functions, and are the inverses of each other.*

We also show that a conjugate pair of closed proper M-convex and L-convex functions arise from the minimum cost flow/tension problems.

The organization of this paper is as follows. In Section 2 we provide the precise definitions of M-/M$^{\natural}$-convex and L-/L$^{\natural}$-convex functions, and show various examples of these functions. The conjugacy relationship between M-/L-convexity is proven in Section 3.

In this paper, we focus on the conjugacy relationship between M-convex and L-convex functions. See [17] for other properties of M-convex and L-convex functions in continuous variables.

2. M-convex and L-convex Functions over the Real Space

2.1. Definitions of M-convex and L-convex Functions

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ be a function. A function f is said to be *convex* if its epigraph $\{(x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x)\}$ is a convex set. A convex function f with $f > -\infty$ is said to be *proper* if $\mathrm{dom}\, f \neq \emptyset$, and *closed* if its epigraph is a closed set. We denote by $\arg\min f$ the set of minimizers of f, i.e., $\arg\min f = \{x \in \mathbf{R}^n \mid f(x) \leq f(y)\ (\forall y \in \mathbf{R}^n)\}$, which can be the empty set.

Proposition 2.1. *For a closed proper convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$*, any level set* $\{x \in \mathbf{R}^n \mid f(x) \leq \eta\}$ $(\eta \in \mathbf{R})$ *is a closed set, and* $\arg\min f \neq \emptyset$ *if* $\mathrm{dom}\, f$ *is bounded.*

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a convex function and $x \in \mathrm{dom}\, f$. The subdifferential of f at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) = \{p \in \mathbf{R}^n \mid f(y) \geq f(x) + \langle p, y - x\rangle\ (\forall y \in \mathbf{R}^n)\}.$$

For $d \in \mathbf{R}^n$, the directional derivative of f at x w.r.t. d is defined by

$$f'(x; d) = \lim_{\alpha \downarrow 0} \{f(x + \alpha d) - f(x)\}/\alpha.$$

We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M-convex* if it is convex and satisfies (M-EXC); we say that f is *closed proper M-convex* if it is closed proper convex, in addition. The effective domain $\mathrm{dom}\, f$ of a closed proper M-convex function f is contained in a hyperplane $\{x \in \mathbf{R}^n \mid x(N) = r\}$ for some $r \in \mathbf{R}$, where $x(N) = \sum_{i=1}^{n} x(i)$.

Proposition 2.2. *If f is closed proper M-convex, then $x(N) = y(N)$ for all $x, y \in \mathrm{dom}\, f$.*

Proof. To the contrary assume $x(N) > y(N)$ for some $x, y \in \mathrm{dom}\, f$. Put

$$S = \{z \in \mathbf{R}^n \mid x \wedge y \leq z \leq x \vee y,\ f(z) \leq \max\{f(x), f(y)\}\},$$

which is a bounded closed set. Let $x_*, y_* \in S$ minimize the value $||x_* - y_*||_1$ among all pairs of vectors in S with $x_*(N) = x(N)$ and $y_*(N) = y(N)$. The property (M-EXC) for x_* and y_* implies

$$2\max\{f(x), f(y)\} \geq f(x_*)+f(y_*) \geq f(x_*-\alpha(\chi_i-\chi_j))+f(y_*+\alpha(\chi_i-\chi_j)) \quad (2.1)$$

for some $i \in \mathrm{supp}^+(x_* - y_*)$, $j \in \mathrm{supp}^-(x_* - y_*)$, and a sufficiently small $\alpha > 0$. Putting $\widehat{x} = x_* - \alpha(\chi_i - \chi_j)$ and $\widehat{y} = y_* + \alpha(\chi_i - \chi_j)$, we have $\widehat{x}(N) = x_*(N)$, $\widehat{y}(N) = y_*(N)$. Moreover, (2.1) implies $\widehat{x} \in S$ or $\widehat{y} \in S$, a contradiction to the choice of x_* and y_* since $||\widehat{x} - y_*||_1 < ||x_* - y_*||_1$ and $||x_* - \widehat{y}||_1 < ||x_* - y_*||_1$. □

Hence, a closed proper M-convex function loses no information other than r when projected onto an $(n-1)$-dimensional space. We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M$^\natural$-convex* if the function $\widehat{f} : \mathbf{R}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x, x_{n+1}) = \begin{cases} f(x) & (x \in \mathbf{R}^n,\ x_{n+1} \in \mathbf{R},\ x_{n+1} = -x(N)), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex; we say that f is *closed proper M$^\natural$-convex* if it is closed proper convex, in addition.

Remark 2.3. The property (M-EXC) alone is independent of good properties such as convexity and continuity. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a function satisfying so-called *Jensen's equation*:

$$\varphi(\alpha) + \varphi(\beta) = 2\varphi((\alpha + \beta)/2) \qquad (\forall \alpha, \beta \in \mathbf{R}). \quad (2.2)$$

It is known that there are discontinuous and nonconvex functions satisfying Jensen's equation (see, e.g., [1, pp. 43–48], [22, p. 217]). Consider such a function φ, and define $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x(1), x(2)) = \begin{cases} \varphi(x(1)) & ((x(1), x(2)) \in \mathbf{R}^2,\ x(1) + x(2) = 0), \\ +\infty & (\text{otherwise}). \end{cases} \quad (2.3)$$

Then, (M-EXC) for f follows immediately from Jensen's equation for φ; however, f is neither convex nor continuous. □

On the other hand, we call a function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *L-convex* if g is a convex function satisfying (LF1) and (LF2); we say that g is *closed proper L-convex* if it is closed proper convex, in addition. Due to the property (LF2), an L-convex function loses no information other than r when restricted to a hyperplane $\{p \in \mathbf{R}^n \mid p(i) = 0\}$ for any $i \in N$. We call a function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *$L^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p, p_{n+1}) = g(p - p_{n+1}\mathbf{1}) \quad (p \in \mathbf{R}^n,\ p_{n+1} \in \mathbf{R})$$

is L-convex; we say that g is *closed proper $L^\natural$-convex* if it is closed proper convex, in addition.

Remark 2.4. The properties (LF1) and (LF2) are independent of convexity and continuity. Consider a function $\psi : \mathbf{R} \to \mathbf{R}$ satisfying Jensen's equation (2.2) such that ψ is neither continuous nor convex. Define a function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ as

$$g(p(1), p(2)) = \psi(p(1) - p(2)) \quad ((p(1), p(2)) \in \mathbf{R}^2). \tag{2.4}$$

Then, g satisfies the submodular inequality (LF1) with equality and (LF2) with $r = 0$, and is neither convex nor continuous. □

In the following discussion, we mainly consider the classes of closed proper M-/L-/M$^\natural$-/L$^\natural$-convex functions. We denote by $\mathcal{M}_n$ (resp. $\mathcal{L}_n$) the class of closed proper M-convex (resp. L-convex) functions in n variables:

$$\mathcal{M}_n = \{f \mid f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{closed proper M-convex}\},$$
$$\mathcal{L}_n = \{g \mid g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{closed proper L-convex}\}.$$

We define $\mathcal{M}_n^\natural$ and $\mathcal{L}_n^\natural$ to be the classes of closed proper M$^\natural$-convex and L$^\natural$-convex functions, respectively. As is obvious from the definitions, closed proper M$^\natural$-convex (resp. L$^\natural$-convex) function is essentially equivalent to closed proper M-convex (resp. L-convex) function, whereas the class of closed proper M$^\natural$-convex (resp. L$^\natural$-convex) functions contains that of closed proper M-convex (resp. L-convex) functions as a proper subclass. These relationships can be summarized as

$$\mathcal{M}_n \subset \mathcal{M}_n^\natural \simeq \mathcal{M}_{n+1}, \qquad \mathcal{L}_n \subset \mathcal{L}_n^\natural \simeq \mathcal{L}_{n+1}.$$

2.2. Examples

M-/M$^\natural$-convex and L-/L$^\natural$-convex functions have rich examples [12,13,15,16].

Example 2.5 (affine functions). For $p_0 \in \mathbf{R}^n$ and $\beta \in \mathbf{R}$, the function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by $f(x) = \langle p_0, x\rangle + \beta$ ($x \in \mathrm{dom}\, f$) is closed proper M-convex or closed proper M$^\natural$-convex according as $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = 0\}$ or $\mathrm{dom}\, f = \mathbf{R}^n$. For $x_0 \in \mathbf{R}^n$ and $\nu \in \mathbf{R}$, the function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by $g(p) = \langle p, x_0\rangle + \nu$ ($p \in \mathbf{R}^n$) is closed proper L-convex as well as closed proper L$^\natural$-convex. □

We denote by $\mathcal{C}^1$ the class of univariate closed proper convex functions, i.e.,

$$\mathcal{C}^1 = \{\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\} \mid \varphi :\ \text{closed proper convex}\}.$$

Recall that the conjugate function $f^\bullet$ of a function f is defined by (1.2).

Example 2.6. For $\varphi, \psi \in \mathcal{C}^1$, the functions $f, g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ given by (2.3) and (2.4), respectively, are closed proper M-convex and closed proper L-convex, respectively. Moreover, if φ and ψ are conjugate to each other, then f and g are conjugate to each other. □

Example 2.7 (separable-convex functions). Let $f_i \in \mathcal{C}^1$ $(i \in N)$ be a family of univariate closed proper convex functions. The function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \sum_{i=1}^{n} f_i(x(i)) \qquad (x \in \mathbf{R}^n)$$

is closed proper M$^\natural$-convex as well as closed proper L$^\natural$-convex. The restriction $\check{f}$ of f to the hyperplane $\{x \in \mathbf{R}^n \mid x(N) = 0\}$ is closed proper M-convex if its effective domain is nonempty.

For functions $g_{ij} \in \mathcal{C}^1$ indexed by $i, j \in N$, the function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(p(j) - p(i)) \qquad (p \in \mathbf{R}^n)$$

is closed proper L-convex with $r = 0$ in (LF2) if $\mathrm{dom}\, g \neq \emptyset$.

Proof. M$^\natural$-convexity of f and M-convexity of $\check{f}$ follow from the inequality

$$f_i(\alpha) + f_i(\beta) \geq f_i(\alpha + \delta) + f_i(\beta - \delta)$$

for any $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$ and any $\delta \in [0, \beta - \alpha]$, whereas L-convexity of g is by the inequality

$$g_{ij}(\lambda - \mu) + g_{ij}(\lambda' - \mu') \geq g_{ij}(\lambda - \mu') + g_{ij}(\lambda' - \mu)$$

for any $\lambda, \lambda', \mu, \mu' \in \mathbf{R}$ with $\lambda \geq \lambda'$, $\mu \leq \mu'$. L$^\natural$-convexity of f is a special case of L-convexity of g (see the definition of L$^\natural$-convex functions). □

Example 2.8 (quadratic functions). Let $A = (a(i,j))_{i,j=1}^{n} \in \mathbf{R}^{n \times n}$ be a symmetric matrix. Define a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x) = (1/2)x^{\mathrm{T}} A x$ $(x \in \mathbf{R}^n)$. Then, f is M$^\natural$-convex if and only if

$$x^{\mathrm{T}} a_i \geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)),$$

where a_i denotes the i-th column of A for $i \in N$. The function f is L$^\natural$-convex if and only if

$$a(i,j) \leq 0 \quad (\forall i, j \in N,\ i \neq j), \qquad \sum_{i=1}^{n} a(i,j) \geq 0 \quad (\forall j \in N).$$

See [16] for proofs. □

Example 2.9 (minimum cost flow/tension problems). M-/L-convex functions arise from the minimum cost flow/tension problems with nonlinear cost functions.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose that we are given a family of convex functions $f_a \in \mathcal{C}^1$ $(a \in A)$, each of which represents the cost of flow on arc a. A vector $\xi \in \mathbf{R}^A$ is called a flow, and the boundary $\partial\xi \in \mathbf{R}^V$ of a flow ξ is given by

$$\partial\xi(v) = \sum\{\xi(a) \mid \text{ arc } a \text{ leaves } v\} - \sum\{\xi(a) \mid \text{ arc } a \text{ enters } v\} \qquad (v \in V).$$

Then, the minimum cost of a flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is represented by a function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$f(x) = \inf_{\xi}\{\sum_{a \in A} f_a(\xi(a)) \mid (\partial\xi)(v) = -x(v) \ (v \in T),\ (\partial\xi)(v) = 0 \ (v \in V \setminus T)\}.$$

On the other hand, suppose that we are given another family of convex functions $g_a \in \mathcal{C}^1$ $(a \in A)$, each of which represents the cost of tension on arc a. Any vector $\widetilde{p} \in \mathbf{R}^V$ is called a potential, and the coboundary $\delta\widetilde{p} \in \mathbf{R}^A$ of a potential $\widetilde{p}$ is defined by $\delta\widetilde{p}(a) = \widetilde{p}(u) - \widetilde{p}(v)$ for $a = (u, v) \in A$. Then, the minimum cost of a tension that realizes a potential vector $p \in \mathbf{R}^T$ is represented by a function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$g(p) = \inf_{\eta, \widetilde{p}}\{\sum_{a \in A} g_a(\eta(a)) \mid \eta(a) = -\delta\widetilde{p}(a) \ (a \in A),\ \widetilde{p}(v) = p(v) \ (v \in T)\}.$$

It can be shown that both f and g are closed proper convex if $f(x_0)$ and $g(p_0)$ are finite for some $x_0 \in \mathbf{R}^T$ and $p_0 \in \mathbf{R}^T$, which is a direct extension of the results in Iri [6] and Rockafellar [21] for the case of $|T| = 2$. These functions, however, are equipped with different combinatorial structures; f is M-convex and g is L-convex, as follows.

Theorem 2.10. *If f_a and g_a are conjugate to each other for all $a \in A$, then f and g are closed proper M-convex and closed proper L-convex, respectively, and conjugate to each other, where it is assumed that at least one of the following conditions holds:*

(a) $-\infty < f(x_0) < +\infty$ *for some* $x_0 \in \mathbf{R}^T$,
(b) $-\infty < g(p_0) < +\infty$ *for some* $p_0 \in \mathbf{R}^T$,
(c) $f(x_0) < +\infty, g(p_0) < +\infty$ *for some* $x_0 \in \mathbf{R}^T, p_0 \in \mathbf{R}^T$.

We first prove the closedness of f and g and the conjugacy relationship. For this, we use the following duality theorem for the minimum cost flow/tension problems.

Theorem 2.11 (cf. [21, Sec. 8H]). *Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Also, let $f_a, g_a \in \mathcal{C}^1$ $(a \in A)$ and $f_v, g_v \in \mathcal{C}^1$*

$(v \in T)$ be conjugate pairs of closed convex functions. Then, we have

$$\inf_{\xi, x} \left\{ \sum_{a \in A} f_a(\xi(a)) + \sum_{v \in T} f_v(x(v)) \;\middle|\; \begin{array}{l} (\partial\xi)(v) = -x(v) \ (v \in T), \\ (\partial\xi)(v) = 0 \ (v \in V \setminus T) \end{array} \right\}$$
$$= \sup_{\eta, \tilde{p}} \left\{ -\sum_{a \in A} g_a(\eta(a)) - \sum_{v \in T} g_v(-\widetilde{p}(v)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a) \ (a \in A) \right\}$$

unless $\inf = +\infty$ *and* $\sup = -\infty$.

Lemma 2.12. *Let $x \in \mathbf{R}^T$ and $p \in \mathbf{R}^T$.*
(i) *$f(x) = g^\bullet(x)$ if $f(x) < +\infty$ or $g(p_0) < +\infty$ for some $p_0 \in \mathbf{R}^T$.*
(ii) *$g(p) = f^\bullet(p)$ if $g(p) < +\infty$ or $f(x_0) < +\infty$ for some $x_0 \in \mathbf{R}^T$.*

Proof. To prove (i), consider functions $f_v, g_v \in \mathcal{C}^1$ $(v \in T)$ given as

$$f_v(\alpha) = \begin{cases} 0 & (\alpha = x(v)), \\ +\infty & (\alpha \neq x(v)), \end{cases} \qquad g_v(\beta) = x(v)\beta \quad (\beta \in \mathbf{R})$$

for the given $x \in \mathbf{R}^T$. The functions f_v and g_v are conjugate to each other for each $v \in T$. If $f(x) < +\infty$ or $g(p_0) < +\infty$ for some $p_0 \in \mathbf{R}^T$, then Theorem 2.11 implies that

$$f(x) = \inf_{\xi, x'} \left\{ \sum_{a \in A} f_a(\xi(a)) + \sum_{v \in T} f_v(x'(v)) \;\middle|\; \begin{array}{l} (\partial\xi)(v) = -x'(v) \ (v \in T), \\ (\partial\xi)(v) = 0 \ (v \in V \setminus T) \end{array} \right\}$$
$$= \sup_{\eta, \tilde{p}} \left\{ \sum_{v \in T} \widetilde{p}(v)x(v) - \sum_{a \in A} g_a(\eta(a)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a) \ (a \in A) \right\}$$
$$= \sup\{\langle p, x\rangle - g(p) \mid p \in \mathbf{R}^T\} \ = \ g^\bullet(x).$$

The proof for (ii) is similar to that for (i) and therefore omitted. □

Suppose $f(x_0) < +\infty$ holds for some $x_0 \in \mathbf{R}^T$. By Lemma 2.12 (i) we have $f(x_0) > -\infty$ if and only if $g(p_0) < +\infty$ for some $p_0 \in \mathbf{R}^T$. This shows that the condition (a) is equivalent to (c). We can show the equivalence between (b) and (c) in the similar way by using Lemma 2.12 (ii). Hence, Lemma 2.12 implies that if one of conditions (a), (b) and (c) holds, then we have $f = g^\bullet$ and $g = f^\bullet$. This conjugacy relationship shows that f and g are closed proper convex, in particular.

We then prove the M-convexity of f and the L-convexity of g.
[(M-EXC) for f] Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$. For any $\varepsilon > 0$ and $z \in \{x, y\}$, there exist $\xi_z \in \mathbf{R}^A$ with

$$\sum_{a \in A} f_a(\xi_z(a)) \leq f(z) + \varepsilon, \quad (\partial\xi_z)(v) = -z(v) \ (v \in T), \quad (\partial\xi_z)(v) = 0 \ (v \in V \setminus T).$$

By a standard augmenting path argument, there exist $\pi \in \{0, \pm 1\}^A$ and $v \in \mathrm{supp}^-(x - y)$ $(\subseteq T)$ such that

$$\mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi_y - \xi_x), \ \mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi_y - \xi_x), \ \partial\pi = \chi_u - \chi_v,$$

where we can assume the following inequality with $m = |A|$, $n = |V|$:

$$\min\{|\xi_x(a) - \xi_y(a)| \mid a \in A,\ \pi(a) = \pm 1\} \geq (x(u) - y(u))/m^n.$$

Putting $\alpha_0 = (x(u) - y(u))/m^n$, we have

$$f_a(\xi_x(a) + \alpha\pi(a)) + f_a(\xi_y(a) - \alpha\pi(a)) \leq f_a(\xi_x(a)) + f_a(\xi_y(a)) \quad (\alpha \in [0, \alpha_0])$$

for all $a \in A$. Hence it follows that

$$\begin{aligned} & f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \\ & \leq \sum_{a \in A} [f_a(\xi_x(a) + \alpha\pi(a)) + f_a(\xi_y(a) - \alpha\pi(a))] \\ & \leq \sum_{a \in A} [f_a(\xi_x(a)) + f_a(\xi_y(a))] \ \leq \ f(x) + f(y) + 2\varepsilon \qquad (\alpha \in [0, \alpha_0]). \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily and T is a finite set, there exists some $v = v_*$ satisfying

$$f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \leq f(x) + f(y) \qquad (\alpha \in [0, \alpha_0]),$$

implying (M-EXC) for f.

[L-convexity for g] Let $p, q \in \mathrm{dom}\, g$. For any $\varepsilon > 0$ there exist $\widetilde{p}, \widetilde{q} \in \mathbf{R}^V$ with

$$\begin{aligned} & \sum_{a \in A} g_a(-\delta\widetilde{p}(a)) \leq g(p) + \varepsilon, \quad \widetilde{p}(v) = p(v) \ (v \in T), \\ & \sum_{a \in A} g_a(-\delta\widetilde{q}(a)) \leq g(q) + \varepsilon, \quad \widetilde{q}(v) = q(v) \ (v \in T). \end{aligned}$$

It holds that $(\widetilde{p} \wedge \widetilde{q})(v) = (p \wedge q)(v)$, $(\widetilde{p} \vee \widetilde{q})(v) = (p \vee q)(v)$ for all $v \in T$, and

$$g_a(-\delta(\widetilde{p} \wedge \widetilde{q})(a)) + g_a(-\delta(\widetilde{p} \vee \widetilde{q})(a)) \leq g_a(-\delta\widetilde{p}(a)) + g_a(-\delta\widetilde{q}(a)) \qquad (a \in A)$$

by convexity of g_a. Hence it follows that

$$\begin{aligned} g(p \wedge q) + g(p \vee q) & \leq \sum_{a \in A} g_a(-\delta(\widetilde{p} \wedge \widetilde{q})(a)) + \sum_{a \in A} g_a(-\delta(\widetilde{p} \vee \widetilde{q})(a)) \\ & \leq \sum_{a \in A} g_a(-\delta\widetilde{p}(a)) + \sum_{a \in A} g_a(-\delta\widetilde{q}(a)) \ \leq \ g(p) + g(q) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily, we have $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q)$, implying (LF1) for g. The property (LF2) for g is immediate from the equation $\delta(\widetilde{p} + \lambda\mathbf{1}) = \delta\widetilde{p}$ for $\widetilde{p} \in \mathbf{R}^V$ and $\lambda \in \mathbf{R}$. □

3. Proof of Conjugacy Relationship

In this section, we prove Theorem 1.1, the main result of this paper.

The conjugacy operation $f \mapsto f^\bullet$ given by (1.2) induces a symmetric one-to-one correspondence in the class of all closed proper convex functions on $\mathbf{R}^n$.

Theorem 3.1 ([20, Th. 12.2]). *For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, its conjugate $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is also a closed proper convex function with $f^{\bullet\bullet} = f$.*

By Theorem 3.1, it remains to show that "$f \in \mathcal{M}_n \Longrightarrow f^\bullet \in \mathcal{L}_n$" and "$g \in \mathcal{L}_n \Longrightarrow g^\bullet \in \mathcal{M}_n$."

3.1. Proof of "$f \in \mathcal{M}_n \Longrightarrow f^\bullet \in \mathcal{L}_n$"

Let $f \in \mathcal{M}_n$. To prove (LF2) for $f^\bullet$, we put $r = x(N)$ with some $x \in \mathrm{dom}\, f$, which is independent of the choice of x by Proposition 2.2. For $p \in \mathrm{dom}\, f^\bullet$ and $\lambda \in \mathbf{R}$, we have

$$\begin{aligned} f^\bullet(p + \lambda \mathbf{1}) &= \sup\{\langle p + \lambda \mathbf{1}, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\} \\ &= \sup\{\langle p, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\} + \lambda x(N) \ = \ f^\bullet(p) + \lambda r, \end{aligned}$$

implying (LF2) for $f^\bullet$.

To prove the submodularity (LF1) for $f^\bullet$, we may assume that $\mathrm{dom}\, f$ is bounded. The case where $\mathrm{dom}\, f$ is unbounded can be reduced to the case of bounded $\mathrm{dom}\, f$ as follows. For a fixed vector $x_0 \in \mathrm{dom}\, f$, we define $f_k : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ by

$$f_k(x) = \begin{cases} f(x) & (\text{if } \|x - x_0\|_\infty \le k), \\ +\infty & (\text{otherwise}). \end{cases}$$

Since $f_k \in \mathcal{M}_n$ and $\mathrm{dom}\, f_k$ is bounded, $f_k^\bullet$ fulfills (LF1).

Let $p \in \mathrm{dom}\, f^\bullet$. By the definition of f_k, it is easy to see that $f^\bullet(p) \ge f_k^\bullet(p)$. For any $\varepsilon > 0$, there exists $x_\varepsilon \in \mathrm{dom}\, f$ with $f^\bullet(p) - \varepsilon < \langle p, x_\varepsilon\rangle - f(x_\varepsilon)$, implying $f^\bullet(p) - \varepsilon \le f_k^\bullet(p)$ for any $k \ge \|x_\varepsilon - x_0\|_\infty$. Therefore, we have $f^\bullet(p) = \lim_{k\to\infty} f_k^\bullet(p)$.

Hence, for any $p, q \in \mathrm{dom}\, f^\bullet$ it holds that

$$\begin{aligned} f^\bullet(p) + f^\bullet(q) &= \lim_{k\to\infty} \{f_k^\bullet(p) + f_k^\bullet(q)\} \\ &\ge \lim_{k\to\infty} \{f_k^\bullet(p \wedge q) + f_k^\bullet(p \vee q)\} = f^\bullet(p \wedge q) + f^\bullet(p \vee q), \end{aligned}$$

i.e., the submodularity (LF1) holds for $f^\bullet$.

We now assume that $\mathrm{dom}\, f$ is bounded and prove the submodularity (LF1) for $f^\bullet$. Since $\mathrm{dom}\, f^\bullet = \mathbf{R}^n$, the submodularity of $f^\bullet$ is equivalent to the local submodularity (see, e.g., [15, Th. 4.27]):

$$f^\bullet(p + \lambda\chi_i) + f^\bullet(p + \mu\chi_j) \ge f^\bullet(p) + f^\bullet(p + \lambda\chi_i + \mu\chi_j), \tag{3.1}$$

where $p \in \mathbf{R}^n$, $i, j \in N$ are distinct indices, and λ, μ are nonnegative reals. The proof of local submodularity (3.1) for $f^\bullet$ consists of the following steps.

1. Fix $p \in \mathbf{R}^n$ and $i, j \in N$, and define functions $\widetilde{g}, \widetilde{f} : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$\widetilde{g}(\lambda, \mu) = f^\bullet(p + \lambda\chi_i + \mu\chi_j) \qquad (\lambda, \mu \in \mathbf{R}), \tag{3.2}$$

$$\widetilde{f}(\alpha, \beta) = \inf_x \{f(x) - \langle p, x\rangle \mid x \in \mathrm{dom}\, f,\ x(i) = \alpha,\ x(j) = \beta\} \qquad (\alpha, \beta \in \mathbf{R}). \tag{3.3}$$

   Then, $\widetilde{g} = (\widetilde{f})^\bullet$ (Lemma 3.2).
2. (M-EXC) for f implies supermodularity of $\widetilde{f}$ (Proposition 3.4, Lemma 3.5).
3. Supermodularity of $\widetilde{f}$ implies submodularity of $(\widetilde{f})^\bullet$ (Proposition 3.6).
4. Submodularity of $\widetilde{g}$ immediately implies local submodularity (3.1).

Lemma 3.2. *The functions $\widetilde{g}$ and $\widetilde{f}$ given by (3.2) and (3.3), respectively, satisfy $\widetilde{g} = (\widetilde{f})^\bullet$.*

Proof.

$$\begin{aligned}(\widetilde{f})^\bullet(\lambda, \mu) &= \sup\{\lambda\alpha + \mu\beta - \widetilde{f}(\alpha, \beta) \mid \alpha, \beta \in \mathbf{R}\}\\ &= \sup\{\lambda x(i) + \mu x(j) + \langle p, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\}\\ &= f^\bullet(p + \lambda\chi_i + \mu\chi_j) \ = \ \widetilde{g}(\lambda, \mu).\end{aligned}$$

□

Proposition 3.3 ([20, Cor. 7.5.1]). *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper convex function. For $x \in \mathbf{R}^n$ and $y \in \mathrm{dom}\, f$, we have*

$$f(x) = \lim_{\lambda \uparrow 1} f(\lambda x + (1 - \lambda)y).$$

Proposition 3.4. *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper convex function satisfying the property:*

(M-P0) $\forall x, y \in \mathrm{dom}\, f$ *with* $x \geq y$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists \alpha_0 > 0$*:*

$$f(x) + f(y) \geq f(x - \alpha\chi_i) + f(y + \alpha\chi_i) \qquad (\alpha \in [0, \alpha_0]).$$

Then, f satisfies the supermodular inequality:

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \qquad (x, y \in \mathbf{R}^n). \tag{3.4}$$

In particular, a closed proper M$^\natural$-convex function satisfies the supermodular inequality (3.4).

Proof. Note that an M$^\natural$-convex function satisfies the property (M-P0). Hence, it suffices to show the former claim only.

To show the supermodularity of f, we first prove that (M-P0) implies the following stronger property:

(M-P1) $\forall x, y \in \mathrm{dom}\, f$ with $x \geq y$, $\forall i \in \mathrm{supp}^+(x - y)$:

$$f(x) + f(y) \geq f(x - (x(i) - y(i))\chi_i) + f(y + (x(i) - y(i))\chi_i). \tag{3.5}$$

Put $\overline{\alpha} = x(i) - y(i)$, and define functions $\varphi_x, \varphi_y : [0, \overline{\alpha}] \to \mathbf{R} \cup \{+\infty\}$ by

$$\varphi_x(\alpha) = f(x - \alpha\chi_i), \quad \varphi_y(\alpha) = f(y + (\overline{\alpha} - \alpha)\chi_i) \quad (\alpha \in [0, \overline{\alpha}]).$$

Claim 1. Let $\alpha \in [0, \overline{\alpha}]$.
(i) If $\varphi_x(\alpha) < +\infty$, then $\varphi_x((\alpha + \overline{\alpha})/2) < +\infty$.
(ii) If $\varphi_y(\alpha) < +\infty$, then $\varphi_y(\alpha/2) < +\infty$.

[Proof of Claim 1] We prove (i) only, where we may assume $\alpha < \overline{\alpha}$. Put $\widehat{x} = x - \alpha\chi_i$, and

$$\alpha_* = \sup\{\beta \mid 0 \leq \beta \leq (\overline{\alpha} - \alpha)/2,\ f(\widehat{x} - \beta\chi_i) + f(y + \beta\chi_i) \leq f(\widehat{x}) + f(y)\}.$$

The value $f(\widehat{x} - \beta\chi_i) + f(y + \beta\chi_i)$ is a proper closed convex function in β. Hence, Propositions 2.1 and 3.3 imply that the value α_* is well-defined and satisfies

$$f(\widehat{x} - \alpha_*\chi_i) + f(y + \alpha_*\chi_i) \leq f(\widehat{x}) + f(y). \tag{3.6}$$

Assume, to the contrary, that $\alpha_* < (\overline{\alpha} - \alpha)/2$ $(= (\widehat{x}(i) - y(i))/2)$. Then, we have $i \in \mathrm{supp}^+((\widehat{x} - \alpha_*\chi_i) - (y + \alpha_*\chi_i))$, and therefore (M-P0) for $\widehat{x} - \alpha_*\chi_i$ and $y + \alpha_*\chi_i$ implies that there exists a sufficiently small $\beta > 0$ satisfying

$$f(\widehat{x} - \alpha_*\chi_i) + f(y + \alpha_*\chi_i) \geq f(\widehat{x} - \alpha_*\chi_i - \beta\chi_i) + f(y + \alpha_*\chi_i + \beta\chi_i).$$

From this inequality and (3.6) follows

$$f(\widehat{x}) + f(y) \geq f(\widehat{x} - (\alpha_* + \beta)\chi_i) + f(y + (\alpha_* + \beta)\chi_i),$$

a contradiction to the choice of α_*. Hence, we have $\alpha_* = (\overline{\alpha} - \alpha)/2$. By (3.6), it holds that $\varphi_x((\alpha + \overline{\alpha})/2) = \varphi_x(\alpha + \alpha_*) = f(\widehat{x} - \alpha_*\chi_i) < +\infty$. [End of Claim 1]

Since φ_x and φ_y are convex functions with $\varphi_x(0) < +\infty$, $\varphi_y(\overline{\alpha}) < +\infty$, repeated application of Claim 1 yields

$$\varphi_x(\alpha) < +\infty \quad (0 \leq \forall\alpha < \overline{\alpha}), \qquad \varphi_y(\alpha) < +\infty \quad (0 < \forall\alpha \leq \overline{\alpha}). \tag{3.7}$$

We then define a function $\varphi : [0, \overline{\alpha}] \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\varphi(\alpha) = \varphi_x(\alpha) - \varphi_y(\alpha) \qquad (\alpha \in [0, \overline{\alpha}]).$$

By Proposition 3.3 and (3.7), φ is continuous in the interval $\{\alpha \mid 0 < \alpha < \overline{\alpha}\}$, and satisfies

$$\varphi(0) = \lim_{\alpha \downarrow 0} \varphi(\alpha), \qquad \varphi(\overline{\alpha}) = \lim_{\alpha \uparrow \overline{\alpha}} \varphi(\alpha). \tag{3.8}$$

Claim 2. $\varphi'(\alpha; 1) \leq 0$, $\varphi'(\alpha; -1) \geq 0$ $(0 < \forall\alpha < \overline{\alpha})$.

[Proof of Claim 2] By (3.7) and the convexity of φ_x and φ_y, the directional derivatives $\varphi'(\alpha; \pm 1) = \varphi_x'(\alpha; \pm 1) - \varphi_y'(\alpha; \pm 1)$ are well-defined for all α with $0 < \alpha < \overline{\alpha}$. We here prove $\varphi'(\alpha; 1) \leq 0$ only since $\varphi'(\alpha; -1) \geq 0$ can be proven similarly. Putting $x' = x - \alpha\chi_i$ and $y'_\delta = y + \{\overline{\alpha} - \alpha - \delta\}\chi_i$ for a sufficiently small $\delta > 0$, we have $i \in \mathrm{supp}^+(x' - y'_\delta)$. By (M-P0), there exists some $\beta_0 > 0$

such that $f(x') + f(y'_\delta) \geq f(x' - \beta\chi_i) + f(y'_\delta + \beta\chi_i)$ $(\forall\beta \in [0, \beta_0])$, implying $\varphi'_x(\alpha; 1) \leq -\varphi'_y(\alpha + \delta; -1)$. Hence it follows

$$\varphi'_x(\alpha; 1) \leq -\lim_{\delta\downarrow 0} \varphi'_y(\alpha + \delta; -1) = \varphi'_y(\alpha; 1),$$

which shows $\varphi'(\alpha; 1) \leq 0$. [End of Claim 2]

Claim 2 and (3.8) imply that $\varphi(\alpha)$ is nonincreasing w.r.t. α in the interval $[0, \overline{\alpha}]$. Hence, we have

$$f(x) - f(y + \overline{\alpha}\chi_i) = \varphi(0) \geq \varphi(\overline{\alpha}) = f(x - \overline{\alpha}\chi_i) - f(y),$$

i.e., (3.5) holds. This shows the property (M-P1).

We now prove the supermodularity of f by using the property (M-P1). The proof is by induction on the cardinality of the sets $\mathrm{supp}^+(x-y)$ and $\mathrm{supp}^-(x-y)$. We may assume $x \wedge y, x \vee y \in \mathrm{dom}\, f$, $|\mathrm{supp}^+(x-y)| \geq 1$, and $|\mathrm{supp}^-(x-y)| \geq 1$ since otherwise the supermodular inequality (3.4) holds immediately.

We first consider the case where $|\mathrm{supp}^+(x-y)| = 1$. Let $i \in N$ be the element with $\mathrm{supp}^+(x-y) = \{i\}$. Putting $x' = x \vee y$ and $y' = x \wedge y$, we have $y = x' - (x'(i) - y'(i))\chi_i$ and $x = y' + (x'(i) - y'(i))\chi_i$. Therefore, the supermodular inequality (3.4) follows immediately from (M-P1). The case where $|\mathrm{supp}^-(x-y)| = 1$ can be dealt with similarly by interchanging the roles of x and y.

We then consider the case where $|\mathrm{supp}^+(x-y)| > 1$ and $|\mathrm{supp}^-(x-y)| > 1$. Let $i \in \mathrm{supp}^+(x-y)$. Then, we have $(x \wedge y) + (x(i) - y(i))\chi_i \in \mathrm{dom}\, f$ by (M-P1), and the induction assumption implies

$$\begin{aligned} f(y) - f(x \wedge y) &\leq f(y + (x(i) - y(i))\chi_i) - f((x \wedge y) + (x(i) - y(i))\chi_i) \\ &\leq f(x \vee y) - f(x). \end{aligned}$$

□

Lemma 3.5. *If $f \in \mathcal{M}_n$ and* $\mathrm{dom}\, f$ *is bounded, then the function $\widetilde{f} : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ given by (3.3) satisfies the property* (M-P0) *in Proposition 3.4.*

Proof. We may assume $p = \mathbf{0}$ since $f(x) - \langle p, x\rangle$ is also M-convex as a function in x and since the claim is shown by using the property (M-EXC) only. It suffices to show that for any $(\alpha, \beta), (\alpha', \beta') \in \mathrm{dom}\, \widetilde{f}$ with $\alpha > \alpha'$ and $\beta \geq \beta'$, there exists $\delta_0 > 0$ satisfying

$$\widetilde{f}(\alpha, \beta) + \widetilde{f}(\alpha', \beta') \geq \widetilde{f}(\alpha - \delta, \beta) + \widetilde{f}(\alpha' + \delta, \beta') \qquad (\forall\delta \in [0, \delta_0]).$$

Since f is a closed proper convex function with bounded effective domain, we can show by using Proposition 2.1 that there exist $x, x' \in \mathrm{dom}\, f$ satisfying $x(i) = \alpha$, $x(j) = \beta$, $\widetilde{f}(\alpha, \beta) = f(x)$, and $x'(i) = \alpha'$, $x'(j) = \beta'$, $\widetilde{f}(\alpha', \beta') = f(x')$, respectively. Since $i \in \mathrm{supp}^+(x - x')$, (M-EXC) for f implies that there exist $k \in \mathrm{supp}^-(x - x')$ and $\delta_0 > 0$ satisfying

$$\begin{aligned} \widetilde{f}(\alpha, \beta) + \widetilde{f}(\alpha', \beta') &= f(x) + f(x') \\ &\geq f(x - \delta(\chi_i - \chi_k)) + f(x' + \delta(\chi_i - \chi_k)) \\ &\geq \widetilde{f}(\alpha - \delta, \beta) + \widetilde{f}(\alpha' + \delta, \beta') \qquad (\forall \delta \in [0, \delta_0]), \end{aligned}$$

where it is noted that $k \neq j$ since $j \notin \mathrm{supp}^-(x - x')$. □

Proposition 3.6. *Let $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ be a function in two variables with $\mathrm{dom}\, f \neq \emptyset$. If f is supermodular, then its conjugate $f^\bullet : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ is submodular.*

Proof. It suffices to show

$$f^\bullet(\lambda, \mu) + f^\bullet(\lambda', \mu') \leq f^\bullet(\lambda, \mu') + f^\bullet(\lambda', \mu) \tag{3.9}$$

for $(\lambda, \mu), (\lambda', \mu') \in \mathbf{R}^2$ with $\lambda \geq \lambda'$ and $\mu \geq \mu'$. We claim that

$$[\lambda\alpha + \mu\beta - f(\alpha, \beta)] + [\lambda'\alpha' + \mu'\beta' - f(\alpha', \beta')] \leq f^\bullet(\lambda, \mu') + f^\bullet(\lambda', \mu) \tag{3.10}$$

holds for any $(\alpha, \beta), (\alpha', \beta') \in \mathbf{R}^2$. The inequality (3.9) is immediate from (3.10), since the supremum of the left-hand side of (3.10) over (α, β) and (α', β') coincides with the left-hand side of (3.9).

We now prove (3.10). If $\alpha \geq \alpha'$ and $\beta \geq \beta'$, we have $f(\alpha, \beta) + f(\alpha', \beta') \geq f(\alpha, \beta') + f(\alpha', \beta)$ by the supermodularity of f, and therefore

$$\text{LHS of (3.10)} \leq [\lambda\alpha + \mu'\beta' - f(\alpha, \beta')] + [\lambda'\alpha' + \mu\beta - f(\alpha', \beta)] \leq \text{RHS of (3.10)}.$$

If $\alpha \leq \alpha'$, we have $\lambda\alpha + \lambda'\alpha' \leq \lambda\alpha' + \lambda'\alpha$ and therefore

$$\text{LHS of (3.10)} \leq [\lambda\alpha' + \mu'\beta' - f(\alpha', \beta')] + [\lambda'\alpha + \mu\beta - f(\alpha, \beta)] \leq \text{RHS of (3.10)}.$$

We can prove (3.10) similarly for the case of $\beta \leq \beta'$. □

This concludes the proof of (LF1) for $f^\bullet$ when $\mathrm{dom}\, f$ is bounded.

3.2. Proof of "$g \in \mathcal{L}_n \Longrightarrow g^\bullet \in \mathcal{M}_n$"

Let $g \in \mathcal{L}_n$. It is easy to see that the conjugate function $g^\bullet$ satisfies $\mathrm{dom}\, g^\bullet \subseteq \{x \in \mathbf{R}^n \mid x(N) = r\}$, where $r \in \mathbf{R}$ is the value in (LF2) for g.

We firstly consider the case where $\mathrm{dom}\, g^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$, and prove M-convexity for $g^\bullet$. The proof consists of the following steps. For $x \in \mathbf{R}^n$, we define a function $g[-x] : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ by $g[-x](p) = g(p) - \langle p, x \rangle$ $(p \in \mathbf{R}^n)$.

1. L-convexity of g implies the following property (Lemma 3.7):

(L-P0) $\forall x, y \in \mathbf{R}^n$ with $\arg\min g[-x] \neq \emptyset$, $\arg\min g[-y] \neq \emptyset$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:

$$p(j) - p(i) \leq q(j) - q(i) \quad (\forall p \in \arg\min g[-x], \forall q \in \arg\min g[-y]). \tag{3.11}$$

2. (L-P0) for g implies the following property for $f = g^\bullet$ (Lemma 3.9):
(M-EXC′) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:

$$f'(x; j, i) < +\infty, \;\; f'(y; i, j) < +\infty, \;\text{ and }\; f'(x; j, i) + f'(y; i, j) \leq 0,$$

where $f'(x; j, i) = f'(x; \chi_j - \chi_i)$ for $i, j \in N$.
3. (M-EXC′) for f implies (M-EXC) for f (Theorem 3.10).

Lemma 3.7. *Any* $g \in \mathcal{L}_n$ *satisfies the property* (L-P0).

Proof. First we note that $x(N) = y(N) = r$, where $r \in \mathbf{R}$ is the value in (LF2) for g. It is easy to see that we have

$$p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D, \qquad p \in D,\ \lambda \in \mathbf{R} \Longrightarrow p + \lambda \mathbf{1} \in D$$

for $D = \arg\min g[-x]$ and $D = \arg\min g[-y]$. Therefore, the inequality (3.11) can be rewritten as $p(j) \leq q(j)$ $(\forall p \in D_x,\ \forall q \in D_y)$, where

$$D_x = \{p \in \mathbf{R}^n \mid p \in \arg\min g[-x],\ p(i) = 0\},$$
$$D_y = \{p \in \mathbf{R}^n \mid p \in \arg\min g[-y],\ p(i) = 0\}.$$

Assume, to the contrary, that for any $j \in \mathrm{supp}^-(x-y)$, there exists a pair of vectors $p_j \in D_x$, $q_j \in D_y$ such that $p_j(j) > q_j(j)$. Putting

$$p_x = \bigvee\{p_j \mid j \in \mathrm{supp}^-(x-y)\}, \qquad q_y = \bigwedge\{q_j \mid j \in \mathrm{supp}^-(x-y)\},$$

we have $p_x \in D_x$, $q_y \in D_y$, and $\mathrm{supp}^-(x-y) \subseteq \mathrm{supp}^+(p_x - q_y)$. We also put $S^+ = \mathrm{supp}^+(p_x - q_y)$, $\lambda = \min\{p_x(j) - q_y(j) \mid j \in S^+\}$ (> 0). Then, L-convexity of g implies

$$\begin{aligned} g(p_x) + g(q_y) &= g(p_x - \lambda\mathbf{1}) + g(q_y) + \lambda r \\ &\geq g((p_x - \lambda\mathbf{1}) \vee q_y) + g((p_x - \lambda\mathbf{1}) \wedge q_y) + \lambda r \\ &= g((p_x - \lambda\mathbf{1}) \vee q_y) + g(p_x \wedge (q_y + \lambda\mathbf{1})). \end{aligned} \tag{3.12}$$

Since

$$((p_x - \lambda\mathbf{1}) \vee q_y)(j) = \begin{cases} p_x(j) - \lambda & (j \in S^+), \\ q_y(j) & (j \in N \setminus S^+), \end{cases}$$
$$(p_x \wedge (q_y + \lambda\mathbf{1}))(j) = \begin{cases} q_y(j) + \lambda & (j \in S^+), \\ p_x(j) & (j \in N \setminus S^+), \end{cases}$$

we have

$$\begin{aligned}&\langle (p_x - \lambda\mathbf{1}) \vee q_y, x\rangle + \langle p_x \wedge (q_y + \lambda\mathbf{1}), y\rangle - \langle p_x, x\rangle - \langle q_y, y\rangle \\ &= \lambda \sum_{j\in S^+} (y(j) - x(j)) + \sum_{j\in N\setminus S^+} (q_y(j) - p_x(j))(x(j) - y(j)) \\ &\geq \lambda \sum_{j\in S^+} (y(j) - x(j)) \\ &\geq \lambda \sum_{v\in N\setminus\{i\}} (y(j) - x(j)) \ = \ \lambda(x(i) - y(i)) \ > \ 0,\end{aligned} \tag{3.13}$$

where the inequalities follow from $\mathrm{supp}^-(x-y) \subseteq S^+$. From (3.12) and (3.13) follows

$$g[-x]((p_x - \lambda\mathbf{1}) \vee q_y) + g[-y](p_x \wedge (q_y + \lambda\mathbf{1})) < g[-x](p_x) + g[-y](q_y),$$

a contradiction to the fact that $p_x \in \arg\min g[-x]$, $q_y \in \arg\min g[-y]$. □

Proposition 3.8 (cf. [20, Th. 23.4]). *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a closed proper convex function with* $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = r\}$ *for some* $r \in \mathbf{R}$. *Then, for any* $x \in \mathrm{dom}\, f$ *we have* $\partial f(x) \neq \emptyset$ *and* $f'(x; d) = \sup\{\langle p, d\rangle \mid p \in \partial f(x)\}$ $(d \in \mathbf{R}^n)$.

Lemma 3.9. *Let* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a closed proper convex function with* (L-P0)*. If* $\mathrm{dom}\, g^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$ *for some* $r \in \mathbf{R}$*, then* $g^\bullet$ *satisfies* (M-EXC$'$).

Proof. Let $x, y \in \mathrm{dom}\, g^\bullet$ and $i \in \mathrm{supp}^+(x-y)$. Since $\arg\min g[-x] = \partial g^\bullet(x)$ and $\arg\min g[-y] = \partial g^\bullet(y)$ hold, it follows from Proposition 3.8 that $\arg\min g[-x] \neq \emptyset$ and $\arg\min g[-y] \neq \emptyset$. By the property (L-P0), there exists $j \in \mathrm{supp}^-(x-y)$ satisfying (3.11), implying

$$\begin{aligned}&(g^\bullet)'(x; j, i) + (g^\bullet)'(y; i, j) \\ &= \sup\{p(j) - p(i) \mid p \in \arg\min g[-x]\} + \sup\{q(i) - q(j) \mid q \in \arg\min g[-y]\} \\ &\leq 0,\end{aligned}$$

where the equality is by Proposition 3.8. □

Theorem 3.10. *For a closed proper convex function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, (M-EXC) $\iff$ (M-EXC$'$).

Proof. Since the implication "(M-EXC) $\Longrightarrow$ (M-EXC$'$)" is obvious, we prove below the reverse implication.

Let $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x-y)$. Then, there exists some $j \in \mathrm{supp}^-(x-y)$ and $\alpha_0 > 0$ satisfying

$$f'(x - \alpha(\chi_i - \chi_j); j, i) + f'(y + \alpha(\chi_i - \chi_j); i, j) \leq 0 \qquad (\forall \alpha \in [0, \alpha_0]), \tag{3.14}$$

which is shown later. Put $\varphi(\alpha) = f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j))$ $(\alpha \in \mathbf{R})$. Then, the inequality (3.14) can be rewritten as $\varphi'(\alpha; 1) \leq 0$ $(\forall \alpha \in [0, \alpha_0])$, implying that φ is nonincreasing w.r.t. α in the interval $[0, \alpha_0]$. In particular, we have $\varphi(0) \geq \varphi(\alpha)$ $(\forall \alpha \in [0, \alpha_0])$, i.e., the desired inequality (1.1) holds.

We now prove that (3.14) holds for some $j \in \mathrm{supp}^-(x-y)$ and $\alpha_0 > 0$. Put

$$J = \{j \in \mathrm{supp}^-(x-y) \mid f'(x; j, i) < +\infty,\ f'(y; i, j) < +\infty\},$$

which is nonempty by (M-EXC$'$). Then, there exists a sufficiently small $\beta > 0$ such that $x - |J|\beta(\chi_i - \chi_j) \in \mathrm{dom}\, f$, $y + |J|\beta(\chi_i - \chi_j) \in \mathrm{dom}\, f$ for every $j \in J$. Putting

$$x_* = \frac{1}{|J|}\sum_{j\in J}[x - |J|\beta(\chi_i - \chi_j)] = x - |J|\beta\chi_i + \beta\sum_{j\in J}\chi_j,$$
$$y_* = \frac{1}{|J|}\sum_{j\in J}[y + |J|\beta(\chi_i - \chi_j)] = y + |J|\beta\chi_i - \beta\sum_{j\in J}\chi_j,$$

we have $x_*,\ y_* \in \mathrm{dom}\, f$ by the convexity of $\mathrm{dom}\, f$, and $\mathrm{supp}^+(x_* - y_*) = \mathrm{supp}^+(x-y)$, $\mathrm{supp}^-(x_* - y_*) = \mathrm{supp}^-(x-y)$. By (M-EXC$'$) applied to x_*, y_* and $i \in \mathrm{supp}^+(x_* - y_*)$, there exists $j_0 \in \mathrm{supp}^-(x_* - y_*)$ with

$$f'(x_*; j_0, i) < +\infty, \quad f'(y_*; i, j_0) < +\infty, \quad f'(x_*; j_0, i) + f'(y_*; i, j_0) \leq 0. \quad (3.15)$$

Since $f'(x_*; j_0, i) < +\infty$, we have $x' = x_* + \alpha(\chi_{j_0} - \chi_i) \in \mathrm{dom}\, f$ for some $\alpha > 0$. Since $j_0 \in \mathrm{supp}^+(x' - x)$ and $\mathrm{supp}^-(x' - x) = \{i\}$, the property (M-EXC$'$) for x' and x implies $f'(x; j_0, i) < +\infty$. Similarly, we have $f'(y; i, j_0) < +\infty$. This shows $j_0 \in J$.

The inequality (3.15) and the convexity of f imply

$$f'(x_*; i, j_0) + f'(y_*; j_0, i) \geq 0. \quad (3.16)$$

For $\alpha \in [0, \beta/2]$, we put $x_\alpha = x - \alpha(\chi_i - \chi_{j_0}) \in \mathrm{dom}\, f$ and $y_\alpha = y + \alpha(\chi_i - \chi_{j_0}) \in \mathrm{dom}\, f$. The property (M-EXC$'$) implies

$$f'(x_*; i, j_0) + f'(x_\alpha; j_0, i) \leq 0 \quad (3.17)$$

since $j_0 \in \mathrm{supp}^+(x_* - x_\alpha)$ and $\mathrm{supp}^-(x_* - x_\alpha) = \{i\}$. Similarly, we have

$$f'(y_\alpha; i, j_0) + f'(y_*; j_0, i) \leq 0. \quad (3.18)$$

From (3.16), (3.17), and (3.18) follows (3.14) with $j = j_0$ and $\alpha_0 = \beta/2$. □

This concludes the proof of M-convexity of $g^\bullet$ when $\mathrm{dom}\, g^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$.

We then consider the general case, where the following characterization of M-convex functions is used:

Theorem 3.11. *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a closed proper convex function. Then,* f *satisfies* (M-EXC) *if and only if it satisfies* (M-EXC$_\mathrm{s}$)*:*

(M-EXC$_\mathbf{s}$) *$\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:*

$$f(x)+f(y) \geq f(x-\alpha(\chi_i-\chi_j))+f(y+\alpha(\chi_i-\chi_j)) \quad (\forall \alpha \in [0,(x(i)-y(i))/2t]),$$

where $t = |\mathrm{supp}^-(x-y)|$.

The proof of Theorem 3.11 is given later. For fixed $j_0 \in N$ and $q \in \mathrm{dom}\, g$ with $q(j_0)=0$, we define $g_k : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $(k=1,2,\ldots)$ by

$$g_k(p) = \begin{cases} g(p) \text{ (if } |p(i)-p(j_0)-q(i)| \leq k \text{ for all } i \in N), \\ +\infty \text{ (otherwise)}. \end{cases}$$

It can be easily shown that each g_k is an L-convex function with $\mathrm{dom}\, g_k^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$. Therefore, the discussion above shows that each $g_k^\bullet$ is M-convex, and therefore satisfies (M-EXC$_\mathrm{s}$) by Theorem 3.11. For $x, y \in \mathrm{dom}\, g^\bullet$ ($\subseteq \mathrm{dom}\, g_k^\bullet$) and $i \in \mathrm{supp}^+(x-y)$, there exists some $j_k \in \mathrm{supp}^-(x-y)$ such that

$$g_k^\bullet(x) + g_k^\bullet(y) \geq g_k^\bullet(x-\alpha(\chi_i-\chi_{j_k})) + g_k^\bullet(y+\alpha(\chi_i-\chi_{j_k})) \qquad (\forall \alpha \in [0,(x(i)-y(i))/2t])$$

with $t = |\mathrm{supp}^-(x-y)|$. Since $\mathrm{supp}^-(x-y)$ is a finite set, we may assume that $j_k = j_*$ $(k=1,2,\ldots)$ for some $j_* \in \mathrm{supp}^-(x-y)$. Then, for any $\alpha \in [0,(x(i)-y(i))/2t]$ we have

$$\begin{aligned} g^\bullet(x) + g^\bullet(y) &= \lim_{k\to\infty} \{g_k^\bullet(x) + g_k^\bullet(y)\} \\ &\geq \lim_{k\to\infty} \{g_k^\bullet(x-\alpha(\chi_i-\chi_{j_*})) + g_k^\bullet(y+\alpha(\chi_i-\chi_{j_*}))\} \\ &= g^\bullet(x-\alpha(\chi_i-\chi_{j_*})) + g^\bullet(y+\alpha(\chi_i-\chi_{j_*})). \end{aligned}$$

Thus, (M-EXC$_\mathrm{s}$) holds for $g^\bullet$, which shows M-convexity of $g^\bullet$ by Theorem 3.11. This concludes the proof of M-convexity of $g^\bullet$ for the general case.

We now prove Theorem 3.11. Since the implication "(M-EXC$_\mathrm{s}$) $\Longrightarrow$ (M-EXC)" is obvious, we prove below the reverse implication. We use the following property which is a restatement of Proposition 3.4 in terms of M-convex functions.

Proposition 3.12. *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a closed proper M-convex function. For any $x, y \in \mathbf{R}^n$ and $i \in N$ we have $f(x)+f(y) \leq f(\hat{x}) + f(\check{y})$, where $\hat{x}$ and $\check{y}$ are given as*

$$\hat{x}(j) = \begin{cases} \min\{x(j), y(j)\} & (j \in N \setminus \{i\}), \\ x(N) - \sum_{k \in N\setminus\{i\}} \min\{x(k), y(k)\} & (j = i), \end{cases}$$

$$\check{y}(j) = \begin{cases} \max\{x(j), y(j)\} & (j \in N \setminus \{i\}), \\ y(N) - \sum_{k \in N\setminus\{i\}} \max\{x(k), y(k)\} & (j = i). \end{cases}$$

Proof. By Proposition 2.2, there exists $r \in \mathbf{R}$ with $x(N) = r$ for all $x \in \mathrm{dom}\, f$. We put $N' = N \setminus \{i\}$, and define a function $\overline{f} : \mathbf{R}^{N'} \to \mathbf{R} \cup \{+\infty\}$ by $\overline{f}(x') = f(x)$, where $x \in \mathbf{R}^n$ is the vector with $x(N) = r$ given by $x(j) = x'(j)$ for $j \in N'$ and $x(i) = r - \sum_{j \in N'} x'(j)$. Then, $\overline{f}$ is a closed proper M$^\natural$-convex function; recall the definition in Section 2.1. Therefore, Proposition 3.4 for $\overline{f}$ implies the inequality $f(x) + f(y) \le f(\hat{x}) + f(\check{y})$ when $x(N) = y(N) = r$. If $x(N) \ne r$ or $y(N) \ne r$, then we have $f(x) + f(y) = +\infty = f(\hat{x}) + f(\check{y})$. □

Proof of "(M-EXC) $\Longrightarrow$ (M-EXC$_\mathrm{s}$)". Let $x_0, y_0 \in \mathrm{dom}\, f$, and $i \in \mathrm{supp}^+(x_0 - y_0)$. Put $\mathrm{supp}^-(x_0 - y_0) = \{j_1, j_2, \ldots, j_t\}$. For $h = 1, 2, \ldots, t$, we recursively define a function $\varphi_h : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, $\alpha_h \in \mathbf{R}$, and $x_h, y_h \in \mathbf{R}^n$ by

$$\begin{aligned} &\varphi_h(\alpha) = f(x_{h-1} - \alpha(\chi_i - \chi_{j_h})) + f(y_{h-1} + \alpha(\chi_i - \chi_{j_h})) \qquad (\alpha \in \mathbf{R}),\\ &\alpha_h \;=\; \sup\{\alpha \mid \varphi_h(\alpha) \le \varphi_h(0),\\ &\qquad\qquad \alpha \le \min[x_{h-1}(i) - y_{h-1}(i), y_{h-1}(j_h) - x_{h-1}(j_h)]/2\},\\ &x_h = x_{h-1} - \alpha_h(\chi_i - \chi_{j_h}),\ \ y_h = y_{h-1} + \alpha_h(\chi_i - \chi_{j_h}). \end{aligned}$$

Since each φ_h is closed proper convex, Propositions 2.1 and 3.3 imply that α_h is well-defined and satisfies

$$\begin{aligned} f(x_h) + f(y_h) = \varphi_h(\alpha_h) &= \lim_{\alpha \to \alpha_h} \varphi_h(\alpha)\\ &\le \varphi_h(0) = f(x_{h-1}) + f(y_{h-1}) \quad (h = 1, 2, \ldots, t). \end{aligned} \tag{3.19}$$

We have $x_h, y_h \in \mathrm{dom}\, f$, in particular. Assume, to the contrary, that $\sum_{h=1}^{t} \alpha_h < (x_0(i) - y_0(i))/2$. Since $i \in \mathrm{supp}^+(x_t - y_t)$, there exist some $j_h \in \mathrm{supp}^-(x_t - y_t) \subseteq \mathrm{supp}^-(x_0 - y_0)$ and a sufficiently small $\alpha > 0$ such that

$$f(x_t) + f(y_t) \ge f(x_t - \alpha(\chi_i - \chi_{j_h})) + f(y_t + \alpha(\chi_i - \chi_{j_h})). \tag{3.20}$$

Putting $\widetilde{x}_h = x_h - \alpha(\chi_i - \chi_{j_h})$ and $\widetilde{x}_t = x_t - \alpha(\chi_i - \chi_{j_h})$, we have

$$x_h(k) = \min\{\widetilde{x}_h(k), x_t(k)\}, \qquad \widetilde{x}_t(k) = \max\{\widetilde{x}_h(k), x_t(k)\} \qquad (\forall k \in N \setminus \{i\}).$$

Therefore, Proposition 3.12 implies

$$f(x_h - \alpha(\chi_i - \chi_{j_h})) + f(x_t) \le f(x_h) + f(x_t - \alpha(\chi_i - \chi_{j_h})). \tag{3.21}$$

Similarly, we have

$$f(y_h + \alpha(\chi_i - \chi_{j_h})) + f(y_t) \le f(y_h) + f(y_t + \alpha(\chi_i - \chi_{j_h})). \tag{3.22}$$

From (3.20), (3.21), and (3.22) follows

$$f(x_h - \alpha(\chi_i - \chi_{j_h})) + f(y_h + \alpha(\chi_i - \chi_{j_h})) \le f(x_h) + f(y_h),$$

a contradiction to the definition of x_h and y_h. Hence, we have $\sum_{h=1}^{t} \alpha_h = (x_0(i) - y_0(i))/2$. Let s be the index with $\alpha_s = \max\{\alpha_h \mid 1 \le h \le t\}$. For $\alpha \in [0, \alpha_s]$, we have

$$\begin{aligned} &[f(x_0 - \alpha(\chi_i - \chi_{j_s})) - f(x_0)] + [f(y_0 + \alpha(\chi_i - \chi_{j_s})) - f(y_0)]\\ &\le [f(x_{s-1} - \alpha(\chi_i - \chi_{j_s})) - f(x_{s-1})] + [f(y_{s-1} + \alpha(\chi_i - \chi_{j_s})) - f(y_{s-1})]\\ &\le 0, \end{aligned}$$

where the first inequality is by Proposition 3.12 and the second by (3.19) and convexity of f. This shows (M-EXC$_\mathrm{s}$) for f since $\alpha_s \ge (x_0(i) - y_0(i))/2t$. □

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