NOTE ON THE CONTINUITY OF M-CONVEX AND L-CONVEX
FUNCTIONS IN CONTINUOUS VARIABLES

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Abstract  M-convex and L-convex functions in continuous variables constitute subclasses of convex functions with nice combinatorial properties. In this note we give proofs of the fundamental facts that closed proper M-convex and L-convex functions are continuous on their effective domains.

Keywords: combinatorial optimization, convex function, continuity, submodular function, matroid.

1. Introduction

Two kinds of convexity concepts, called M-convexity and L-convexity, play primary roles in the theory of discrete convex analysis [6]. They are originally introduced for functions in integer variables by Murota [4, 5], and then for functions in continuous variables by Murota–Shioura [8, 10].

M-convex and L-convex functions in continuous variables constitute subclasses of convex functions with additional combinatorial properties such as submodularity and diagonal dominance (see, e.g., [6, 7, 8, 9, 10, 11]). Fundamental properties of M-convex and L-convex functions are investigated in [9], such as equivalent axioms, subgradients, directional derivatives, etc. Conjugacy relationship between M-convex and L-convex functions under the Legendre-Fenchel transformation is shown in [10]. Subclasses of M-convex and L-convex functions are investigated in [8] (polyhedral M-convex and L-convex functions) and in [11] (quadratic M-convex and L-convex functions). As variants of M-convex and L-convex functions, the concepts of $M^2$-convex and $L^2$-convex functions are also introduced by Murota–Shioura [8, 10], where “$M^2$” and “$L^2$” should be read “M-natural” and “L-natural,” respectively.

M-convex and L-convex functions in continuous variables appear naturally in various research areas. In inventory theory, a recent paper of Zipkin [13] sheds a new light on some classical results of Karlin–Scarf [2] and Morton [3] by pointing out that the optimal-cost function possesses $L^2$-convexity. Quadratic $L^2$-convex functions are exactly the same as the (finite dimensional case of) Dirichlet forms used in probability theory [1]. It is shown in [7, Section 14.8] that for (the finite dimensional distribution of) stochastic processes such as Gaussian processes and additive processes, cumulant generating functions and rate functions are $M^2$-convex and $L^2$-convex, respectively. The energy consumed in a nonlinear electrical network is an $L^2$-convex function when expressed as a function in terminal voltages, and is an $M^2$-convex function as a function in terminal currents [6, Section 2.2].
In this note, we discuss continuity issues of M-convex and L-convex functions in continuous variables. Although continuity is one of the most fundamental properties of functions, discussion on continuity is missing in the literature of M-convex and L-convex functions. The aim of this note is to give proofs of the facts that closed proper M-convex and L-convex functions are continuous on their effective domains. The main results of this note are summarized as follows, where the precise definitions of closed proper M-convex and L-convex functions are given in Section 2.1.

**Theorem 1.1.** Let \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \).

(i) If \( f \) is closed proper M-convex, then it is continuous on \( \text{dom } f \).
(ii) If \( f \) is closed proper \( \mathcal{M}^- \)-convex, then it is continuous on \( \text{dom } f \).

**Theorem 1.2.** Let \( g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \).

(i) If \( g \) is closed proper L-convex, then it is continuous on \( \text{dom } g \).
(ii) If \( g \) is closed proper \( L^2 \)-convex, then it is continuous on \( \text{dom } g \).

It may be mentioned that our proof of Theorem 1.2 shows that an L-convex (\( L^2 \)-convex) function is upper semi-continuous even if it is not closed.

2. Preliminaries

2.1. M-convex and L-convex functions

Let \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a function. A function \( f \) is said to be convex if its epigraph \( \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\} \) is a convex set. A convex function \( f \) is said to be proper if the effective domain \( \text{dom } f \) of \( f \) given by \( \text{dom } f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\} \) is nonempty, and closed if its epigraph is a closed set.

A function \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be M-convex if it is convex and satisfies (M-EXC):

\[
(M-EXC) \quad \forall x, y \in \text{dom } f, \forall i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y), \exists \alpha_0 > 0 \text{ satisfying } f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \alpha_0]),
\]

where \( \chi_i \in \{0,1\}^n \) denotes the characteristic vector of \( i \in N = \{1,2,\ldots,n\} \), and

\[
\text{supp}^+(x-y) = \{i \in N \mid x(i) > y(i)\},
\]

\[
\text{supp}^-(x-y) = \{i \in N \mid x(i) < y(i)\}.
\]

We call a function \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) M\(^2\)-convex if the function \( \hat{f}: \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
\hat{f}(x_0, x) = \begin{cases} f(x) & (x_0, x) \in \mathbb{R}^N, \ x_0 = -x(N), \\ +\infty & \text{(otherwise)} \end{cases}
\]

is M-convex, where \( \hat{N} = \{0\} \cup N \) and \( x(N) = \sum_{i \in N} x(i) \). An M-convex (resp., \( \mathcal{M}^- \)-convex) function is said to be closed proper M-convex (resp., closed proper \( \mathcal{M}^- \)-convex) if it is closed and proper, in addition.

A function \( g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be L-convex if it is convex and satisfies (LF1) and (LF2):

\[
(LF1) \quad g(p) + g(q) \geq g(p \land q) + g(p \lor q) \quad (\forall p, q \in \text{dom } g),
\]

\[
(LF2) \quad \exists r \in \mathbb{R}: g(p + \alpha 1) = g(p) + \alpha r \quad (\forall p \in \text{dom } g, \ \forall \alpha \in \mathbb{R}),
\]

where \( p \land q, p \lor q \in \mathbb{R}^n \) are given by

\[
(p \land q)(i) = \min\{p(i), q(i)\}, \quad (p \lor q)(i) = \max\{p(i), q(i)\} \quad (i \in N),
\]
and \( \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n \). We call a function \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) \( L^1 \)-convex if the function \( \hat{g} : \mathbb{R}^\hat{N} \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
\hat{g}(p_0, p) = g(p - p_0 \mathbf{1}) \quad ((p_0, p) \in \mathbb{R}^\hat{N})
\]

is \( L \)-convex, where \( \hat{N} = \{0\} \cup N \). An \( L \)-convex (resp., \( L^1 \)-convex) function is said to be closed proper \( L \)-convex (resp., closed proper \( L^1 \)-convex) if it is closed and proper, in addition.

### 2.2. Basic facts from convex analysis

As technical preliminaries we describe some facts known in convex analysis. This also serves to illustrate the present issue.

Let \( S \) be a subset of \( \mathbb{R}^n \). The affine hull \( \text{aff}(S) \) of \( S \) is given by

\[
\text{aff}(S) = \left\{ \sum_{j=1}^m \lambda_j x_j \mid m : \text{positive integer}, \ x_j \in S, \ \lambda_j \in \mathbb{R} \ (j = 1, 2, \ldots, m), \ \sum_{j=1}^m \lambda_j = 1 \right\}.
\]

We denote by \( \text{cl}(S) \) the closure of \( S \), i.e., the smallest closed set containing \( S \). The relative interior \( \text{ri}(S) \) of \( S \) is given as the set of vectors \( x \in S \) such that there exists a sufficiently small \( \varepsilon > 0 \) satisfying

\[
\{ y \in \mathbb{R}^n \mid \|y - x\| \leq \varepsilon \} \cap \text{aff}(S) \subseteq S.
\]

The relative boundary of \( S \) is given by the set \( \text{cl}(S) \setminus \text{ri}(S) \).

**Theorem 2.1 ([12, Theorem 10.1]).** Any convex function is continuous on the relative interior of the effective domain.

Theorem 2.1 implies, in particular, that a convex function is continuous on the effective domain if the effective domain is an open set.

On the other hand, a convex function is not necessarily continuous at relative boundary points of the effective domain, even if it is closed proper convex, as shown in the following example.

**Example 2.2 ([12, Section 10]).** Let \( f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\} \) be a function defined by

\[
f(x, y) = \begin{cases} 
\frac{y^2}{2x} & (x > 0), \\
0 & (x = y = 0), \\
+\infty & (\text{otherwise}),
\end{cases}
\]

which is closed proper convex since its epigraph \( \{(x, y, z) \in \mathbb{R}^3 \mid z \geq f(x, y)\} \) is a closed convex set. It is easy to see that \( f \) is continuous at every point of \( \text{dom} f \), except at the origin \((x, y) = (0, 0)\). For any positive number \( \alpha \), we have

\[
\lim_{y \to 0} f \left( \frac{y^2}{2\alpha^2}, y \right) = \lim_{y \to 0} \alpha = \alpha \neq 0 = f(0, 0),
\]

which shows that \( f \) is not continuous at the origin.

A sufficient condition for a closed proper convex function to be continuous on the effective domain is given in terms of “locally simplicial” sets. A subset \( S \) of \( \mathbb{R}^n \) is said to be locally simplicial if for each \( x \in S \) there exists a finite collection of simplices \( T_1, T_2, \ldots, T_m \) contained in \( S \) such that

\[
U \cap (T_1 \cup T_2 \cup \cdots \cup T_m) = U \cap S
\]

for some neighborhood \( U \) of \( x \). The class of locally simplicial sets includes line segments, polyhedra, and relatively open convex sets.
Theorem 2.3 ([12, Theorem 10.2]). Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a closed proper convex function. For a locally simplicial set \( S \subseteq \text{dom } f \), the function \( f \) is continuous on \( S \). In particular, \( f \) is continuous on \( \text{dom } f \) if \( \text{dom } f \) is locally simplicial.

3. Continuity of Closed Proper M-/L-convex Functions

We now consider the continuity of closed proper M-/L-convex functions.

The effective domains of closed proper M-/L-convex functions are “essentially polyhedral” in the sense that the closure of the effective domains are polyhedra (see Theorems 3.2 and 3.3 below). Hence, the continuity of closed proper M-/L-convex functions follows from Theorem 2.3 when the effective domains are closed sets. The effective domains of closed proper M-/L-convex functions, however, are not necessarily closed, as shown in the following example.

Example 3.1. Let \( \varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \) be a function defined by

\[
\varphi(x) = \begin{cases} 
\frac{1}{x} & (0 < x \leq 1), \\
+\infty & \text{(otherwise)}.
\end{cases}
\]

Then, \( \varphi \) is a closed proper convex function such that the effective domain \( \text{dom } \varphi \) is an interval \( \{x \in \mathbb{R} \mid 0 < x \leq 1\} \), which is neither a closed set nor a relatively open set.

Using \( \varphi \) we define functions \( f, g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\} \) as follows:

\[
f(x, y) = \begin{cases} 
\varphi(x) & (x + y = 0), \\
+\infty & (x + y \neq 0), 
\end{cases} 
\]

\[
g(x, y) = \varphi(x - y) 
\]

\((x, y) \in \mathbb{R}^2\).

Then, \( f \) and \( g \) are closed proper M-convex and L-convex functions, respectively. Neither \( \text{dom } f \) nor \( \text{dom } g \) is a closed set.

Although the effective domains are not always closed, they are well-behaved and almost polyhedral, as follows.

A polyhedron \( S \subseteq \mathbb{R}^n \) is said to be \( M \)-convex (resp., \( M^2 \)-convex, \( L \)-convex, \( L^2 \)-convex) if the indicator function \( \delta_S : \mathbb{R}^n \to \{0, +\infty\} \) defined by

\[
\delta_S(x) = \begin{cases} 
0 & (x \in S), \\
+\infty & (x \notin S)
\end{cases}
\]

is \( M \)-convex (resp., \( M^2 \)-convex, \( L \)-convex, \( L^2 \)-convex).

Theorem 3.2. For any closed proper \( M \)-convex (resp., \( M^2 \)-convex) function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), the set \( \text{cl}(\text{dom } f) \) is an \( M \)-convex (resp., \( M^2 \)-convex) polyhedron.

Proof. The proof is given in Section 4.1.

Theorem 3.3. For any closed proper \( L \)-convex (resp., \( L^2 \)-convex) function \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), the set \( \text{cl}(\text{dom } g) \) is an \( L \)-convex (resp., \( L^2 \)-convex) polyhedron.

Proof. The proof is given in Section 4.2.

Theorem 3.4. The effective domain of a closed proper \( M \)-convex (resp., \( M^2 \)-convex) function is a locally simplicial set.

Proof. The proof is given in Section 4.3.
Theorem 3.5. The effective domain of a closed proper L-convex (resp., $L^1$-convex) function is a locally simplicial set.

Proof. The proof is given in Section 4.4.

The continuity of closed proper M-/L-convex functions, as claimed in Theorems 1.1 and 1.2, follows from Theorems 2.3, 3.4, and 3.5.

4. Proofs
4.1. Proof of Theorem 3.2
For any closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define a function $f^0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$(f^0^+)(y) = \lim_{\lambda \to \infty} \frac{f(x + \lambda y) - f(x)}{\lambda} \quad (y \in \mathbb{R}^n),$$

where $x \in \mathbb{R}^n$ is any fixed vector in dom $f$. The function $f^0^+$ is called the recession function of $f$ (see [12] for the original definition of the recession function). The recession function $f^0^+$ is a positively homogeneous closed proper convex function, i.e., $f^0^+$ is closed proper convex and satisfies $(f^0^+)(\lambda x) = \lambda(f^0^+)(x)$ for every $x \in \mathbb{R}^n$ and $\lambda > 0$. Our proof of Theorem 3.2 is based on the following fact, where for a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the conjugate $f^*$ of $f$ is given by

$$f^*(p) = \sup\{p^T x - f(x) \mid x \in \text{dom } f\} \quad (p \in \mathbb{R}^n).$$

Theorem 4.1 ([12, Theorem 13.3]). For any closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the recession function $f^0^+$ is the support function of dom $f^*$, i.e., it holds that

$$(f^0^+)(x) = \sup\{p^T x \mid p \in \text{dom } f^*\} \quad (x \in \mathbb{R}^n).$$

It suffices to consider a closed proper M-convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, its conjugate function $g = f^*$ is a closed proper L-convex function [10, Theorem 1.1]. As shown below, the recession function $g^0^+$ of $g$ is L-convex. This implies that the support function of (the closure of) dom $f^*$ is a positively homogeneous L-convex function, which in turn implies that $\text{cl}(\text{dom } f^*)$ is an M-convex polyhedron [8, Theorem 4.38].

We now show the L-convexity of the recession function $g^0^+$. Namely, we prove that $g^0^+$ satisfies (LF1) and (LF2).

Let $p_0 \in \text{dom } g$ be any fixed vector. Then, the recession function $g^0^+$ is given as

$$(g^0^+)(p) = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} \quad (p \in \mathbb{R}^n).$$

Since $g$ satisfies (LF2), there exists $r \in \mathbb{R}$ such that

$$g(p + \alpha \mathbf{1}) = g(p) + \alpha r \quad (\forall p \in \text{dom } g, \forall \alpha \in \mathbb{R}). \quad (4.1)$$

For any $p \in \text{dom } g^0^+$ and $\alpha \in \mathbb{R}$, we have

$$(g^0^+)(p + \alpha \mathbf{1}) = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda(p + \alpha \mathbf{1})) - g(p_0)}{\lambda} = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) + \lambda \alpha r - g(p_0)}{\lambda} = \lim_{\lambda \to \infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} + \alpha r = (g^0^+)(p) + \alpha r,$$
where the second equality is by (4.1). Hence, (LF2) holds for $g^0+$.

Let $p, q \in \text{dom } g^0+$. For any $\lambda \in \mathbb{R}_+$, we have

$$g(p_0 + \lambda p) + g(p_0 + \lambda q) \geq g(p_0 + \lambda (p \wedge q)) + g(p_0 + \lambda (p \vee q))$$

by (LF1) for $g$. Hence, we have

$$g^0+(p) + g^0+(q) = \lim_{\lambda \to -\infty} \frac{g(p_0 + \lambda p) - g(p_0)}{\lambda} + \lim_{\lambda \to -\infty} \frac{g(p_0 + \lambda q) - g(p_0)}{\lambda} \geq \lim_{\lambda \to -\infty} \frac{g(p_0 + \lambda (p \wedge q)) - g(p_0)}{\lambda} + \lim_{\lambda \to -\infty} \frac{g(p_0 + \lambda (p \vee q)) - g(p_0)}{\lambda} = g^0+(p \wedge q) + g^0+(p \vee q),$$

i.e., (LF1) holds for $g^0+$.

### 4.2. Proof of Theorem 3.3

It suffices to consider a closed proper L-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The properties (LF1) and (LF2) for $g$ imply that $D = \text{dom } g$ satisfies the following properties:

1. **(LS1)** $p, q \in D \implies p \wedge q, p \vee q \in D$,
2. **(LS2)** $p \in D \implies p + \lambda 1 \in D$ ($\forall \lambda \in \mathbb{R}$).

Therefore, Theorem 3.3 follows immediately from the next theorem.

#### Theorem 4.2

For any nonempty set $D \subseteq \mathbb{R}^n$, let

$$\gamma_D(i, j) = \sup \{p(j) - p(i) \mid p \in D\}, \quad (i, j \in N),$$

$$\gamma_D = \{p \in \mathbb{R}^n \mid p(j) - p(i) \leq \gamma_D(i, j) \text{ } (i, j \in N)\}.$$

If $D$ satisfies (LS1) and (LS2), then we have $\text{cl}(D) = \gamma_D$.

#### Proof

The inclusion $\text{cl}(D) \subseteq \gamma_D$ is easy to see. To prove the reverse inclusion, we show that $q \in D$ holds for any vector $q$ in the relative interior of $\gamma_D$.

We first show that for any $i, j \in N$ there exists $p_{ij} \in D$ such that

$$p_{ij}(j) - p_{ij}(i) \geq q(j) - q(i).$$

If $-\gamma_D(j, i) = \gamma_D(i, j)$, then any vector in $D$ can be chosen as $p_{ij}$ since for any $p \in D$ we have $p(j) - p(i) = \gamma_D(i, j) = q(j) - q(i)$. Hence, we suppose that $-\gamma_D(j, i) < \gamma_D(i, j)$ holds. Then, we have $q(j) - q(i) \leq \gamma_D(i, j) \text{ since } q \text{ is in the relative interior of } \gamma_D.$

By the definition of $\gamma_D(i, j)$, there exists some $p_{ij} \in D$ such that $q(j) - q(i) \leq p_{ij}(j) - p_{ij}(i) \leq \gamma_D(i, j)$.

By (LS2), we may assume that $p_{ij}(i) = q(i)$ and $p_{ij}(j) \geq q(j)$. For each $i \in N$, the vector $p_i = p_{i1} \lor p_{i2} \lor \cdots \lor p_{in}$ ($\in D$) satisfies $p_i(i) = q(i)$, $p_i(j) \geq q(j)$ for all $j \in N$. Therefore, it holds that $q = p_1 \wedge p_2 \wedge \cdots \wedge p_n \in D$.

#### 4.3. Proof of Theorem 3.4

For any set $S \subseteq \mathbb{R}^n$ and a vector $x \in S$, we denote by $\text{cone}(S, x)$ the conic hull of the vectors $\{y - x \mid y \in S\}$, i.e., $\text{cone}(S, x)$ is the set of vectors $d \in \mathbb{R}^n$ such that $d = \sum_{k=1}^m \alpha_k(y_k - x)$ for some positive integer $m$ and $y_k \in S$, $\alpha_k > 0$ ($k = 1, 2, \ldots, m$). The following is immediate from the definition of locally simplicial sets.

#### Lemma 4.3

A convex set $S \subseteq \mathbb{R}^n$ is locally simplicial if for each $x \in S$, $\text{cone}(S, x)$ is a polyhedral cone.
For the proof of Theorem 3.4 it suffices to consider an M-convex function. Then, Theorem 3.4 follows from Lemma 4.3 and the following lemma.

**Lemma 4.4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a closed proper M-convex function. For any \( x \in \text{dom} \ f \), it holds that

\[
\text{cone}(\text{dom} \ f, x) = \text{cone}(R_x, x),
\]

where \( R_x \subseteq \mathbb{R}^n \) is a polyhedral cone given by

\[
R_x = \{ \chi_j - \chi_i \mid i, j \in N, \ i \neq j, \ x + \alpha(\chi_j - \chi_i) \in \text{dom} \ f \text{ for some } \alpha > 0 \}.
\]

To prove Lemma 4.4 we use the following properties of M-convex functions.

**Lemma 4.5 ([10, Proposition 2.2]).** If \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is closed proper M-convex, then \( x(N) = y(N) \) for all \( x, y \in \text{dom} \ f \).

**Lemma 4.6 ([10, Theorem 3.11]).** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a closed proper convex function. Then, \( f \) satisfies (M-EXC) if and only if it satisfies (M-EXC): (M-EXC) \( \forall x, y \in \text{dom} \ f, \forall i \in \text{supp}^+(x - y), \exists j \in \text{supp}^-(x - y) : \)

\[
f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \alpha_0(x, y, i)]),
\]

where \( \alpha_0(x, y, i) \) is the number given by

\[
\alpha_0(x, y, i) = \frac{x(i) - y(i)}{2|\text{supp}^-(x - y)|}
\]

and satisfies \( \alpha_0(x, y, i) \leq \{y(j) - x(j)\}/2 \).

**Proof of Lemma 4.4.** It is easy to see that \( \text{cone}(R_x, x) \subseteq \text{cone}(\text{dom} \ f, x) \). To prove the reverse inclusion, it suffices to show that \( y - x \in \text{cone}(R_x, x) \) for any \( y \in \text{dom} \ f \).

We will show that there exists a sequence of vectors \( y_k (k = 0, 1, 2, \ldots) \) such that \( y_0 = y \) and

\[
y_k \in \text{dom} \ f, \ y_k \neq x, \ y - y_k \in \text{cone}(R_x, x) \quad (k = 0, 1, 2, \ldots),
\]

\[
||y_{k+1} - x||_1 \leq (1 - \frac{1}{2n^2})||y_k - x||_1.
\]

This implies that \( y - x = \lim_{k \to \infty} (y_k - y_k) \in \text{cone}(R_x, x) \), since \( \text{cone}(R_x, x) \) is a closed set.

We define the vectors \( y_k \ (k = 0, 1, 2, \ldots) \) iteratively as follows. Suppose that \( y_k \) is already defined and satisfies the condition (4.2). Since \( y_k \neq x \), we have \( \text{supp}^+(y_k - x) \neq \emptyset \). Let \( i \in \text{supp}^+(y_k - x) \) be such that

\[
y_k(i) - x(i) = \max\{y_k(i') - x(i') \mid i' \in \text{supp}^+(y_k - x)\}.
\]

By Lemma 4.6, there exists \( j \in \text{supp}^-(y_k - x) \) such that

\[
y_k - \alpha(\chi_i - \chi_j) \in \text{dom} \ f, \ x + \alpha(\chi_i - \chi_j) \in \text{dom} \ f,
\]

where \( \alpha = (y_k(i) - x(i))/2n \). Then, \( y_{k+1} \) is defined as \( y_{k+1} = y_k - \alpha(\chi_i - \chi_j) \).

We now show that the vector \( y_{k+1} \) satisfies the conditions (4.2) and (4.3). Since \( y_{k+1}(i) > x(i) \), we have \( y_{k+1} \neq x \). Since \( x + \alpha(\chi_i - \chi_j) \in \text{dom} \ f \), we have \( \chi_i - \chi_j \in R_x \), which, together with \( y - y_k \in \text{cone}(R_x, x) \), implies

\[
y - y_{k+1} = (y - y_k) + \alpha(\chi_i - \chi_j) \in \text{cone}(R_x, x).
\]
Since \( y_k(N) = x(N) \) by Lemma 4.5, it holds that
\[
\|y_k - x\|_1 = 2 \sum \{ y_k(i') - x(i') \mid i' \in \text{supp}^+(y_k - x) \} \\
\leq 2n(y_k(i) - x(i)) \\
= 4n^2 \alpha,
\]
where the inequality is by (4.4). Hence, it holds that
\[
\|y_{k+1} - x\|_1 = \|y_k - x\|_1 - 2\alpha \leq (1 - \frac{1}{2n^2})\|y_k - x\|_1.
\]

\[\square\]

4.4. Proof of Theorem 3.5

Theorem 3.5 follows from Lemma 4.3 and Lemma 4.7 below. Note that it suffices to consider an \( L \)-convex function.

Lemma 4.7. Let \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be an \( L \)-convex function. For any \( p \in \text{dom} \ g \), it holds that
\[
\text{cone}(\text{dom} \ g, p) = \text{cone}(R_p, p),
\]
where \( R_p \subseteq \mathbb{R}^n \) is a polyhedral cone given by
\[
R_p = \{ \chi_X \mid X \subseteq N, p + \alpha \chi_X \in \text{dom} \ g \text{ for some } \alpha > 0 \} \cup \{+1, -1\}.
\]

To prove Lemma 4.7, we use the following property of \( L \)-convex functions.

Lemma 4.8 ([9, Proposition 3.10]). If \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is \( L \)-convex, then we have
\[
g(p) + g(q) \geq g(p + \lambda \chi X) + g(q - \lambda \chi X)
\]
for all \( p, q \in \text{dom} \ g \) and \( \lambda \in [0, \lambda_1 - \lambda_2] \), where \( \chi X \in \{0, 1\}^n \) denotes the characteristic vector of \( X \subseteq N \), and
\[
\lambda_1 = \max \{ q(i) - p(i) \mid i \in N \}, \\
X = \{ i \in N \mid q(i) - p(i) = \lambda_1 \}, \\
\lambda_2 = \max \{ q(i) - p(i) \mid i \in N \setminus X \}.
\]

Proof of Lemma 4.7. It is easy to see that \( \text{cone}(R_p, p) \subseteq \text{cone}(\text{dom} \ g, p) \), where it is noted that \( p + \alpha \chi \in \text{dom} \ g \) for all \( \alpha \in \mathbb{R} \). To show the reverse inclusion, it suffices to show that \( q - p \in \text{cone}(R_p, p) \) for any \( q \in \text{dom} \ g \).

Since both of the sets \( \text{dom} \ g \) and \( \text{cone}(R_p, p) \) satisfy the property (LS2) (see Section 4.2 for the definition of (LS2)), we may assume that \( p \leq q \) and \( p(i_0) = q(i_0) \) for some \( i_0 \in N \). We prove \( q - p \in \text{cone}(R_p, p) \) by induction on the number \( m \) of distinct values in \( \{ q(i) - p(i) \mid i \in N \} \).

If \( m = 0 \), then we have \( q - p = 0 \in \text{cone}(R_p, p) \). Hence, we assume \( m > 0 \), which implies \( q(i_1) > p(i_1) \) for some \( i_1 \in N \). By Lemma 4.8, we have
\[
p + (\lambda_1 - \lambda_2) \chi X \in \text{dom} \ g, \ q - (\lambda_1 - \lambda_2) \chi X \in \text{dom} \ g,
\]
where
\[
\lambda_1 = \max \{ q(i) - p(i) \mid i \in N \}, \\
X = \{ i \in N \mid q(i) - p(i) = \lambda_1 \}, \\
\lambda_2 = \max \{ q(i) - p(i) \mid i \in N \setminus X \}.
\]
We note that $\lambda_1$ and $\lambda_2$ are finite values and $X$ is a nonempty proper subset of $N$. Put $\widetilde{q} = q - (\lambda_1 - \lambda_2)\chi_X$. Then, the number of distinct values in $\{\widetilde{q}(i) - p(i) \mid i \in N\}$ is equal to $m - 1$. Therefore, the induction hypothesis implies $\widetilde{q} - p \in \text{cone}(R_p, p)$. We also have $\chi_X \in R_p$ since $p + (\lambda_1 - \lambda_2)\chi_X \in \text{dom } g$. Hence, it holds that

$$q - p = (\widetilde{q} - p) + (\lambda_1 - \lambda_2)\chi_X \in \text{cone}(R_p, p).$$

\[\square\]

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