Natural Implementation
in Public Goods Economies

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Abstract

In this paper, we examine what kinds of social choice correspondences (SCCs) are implementable in Nash equilibria by “natural” mechanisms in divisible pure public goods economies. Then, we found conditions for SCCs to be Nash implementable by each of these four natural mechanisms. These characterizations of SCCs depend on the number of goods. First, regardless of the number of goods, the class of Pareto efficient SCCs implemented by $P_2^Q$ mechanisms is equivalent to that of Pareto efficient SCCs implemented by $P^n_Q$ mechanisms. Second, in the case where there exists only one public good, Q, IPQ, $P^2_Q$ and $P^n_Q$ mechanisms implement the Lindahl correspondence for any number of private goods. But, the Pareto correspondence is not implemented by Q, IPQ, $P^2_Q$ or $P^n_Q$ mechanisms, regardless of the number of goods. *Journal of Economic Literature* Classification Numbers: C72, D78, H41.

Key Words: Nash implementation, natural implementation, social choice correspondence, public goods economies.

1 Introduction

In implementation theory, we consider designing a mechanism such that the set of its equilibrium outcomes coincides with that of socially desirable outcomes. A procedure for achieving a socially desirable outcome is represented by a mechanism, which specifies the possible actions available to agents of a society and the outcomes of these actions. Socially desirable outcomes are represented as a social choice correspondence $F$ (abbreviated as SCC $F$). Informally, we say that a mechanism implements a social choice correspondence if for any environment, the set of equilibrium outcomes of the mechanism coincides with the set of outcomes recommended by the social choice correspondence (i.e., $F$-optimal outcomes). If there exists a mechanism that implements a social choice correspondence, the SCC is said to be implementable. Maskin (1977) found the conditions of social choice correspondences to be implementable in Nash equilibria (i.e., Nash implementable) and constructed mechanisms which implement SCCs satisfying these conditions in general environments. Many papers refered to implementation in other equilibria than Nash equilibria. For example, Moore and Repullo (1988) and Abreu and Sen (1990) found the conditions for SCCs to be implementable in subgame perfect equilibria. Palfrey and Srivastava (1991) found the conditions of SCCs to be implementable in undominated Nash equilibria. Jackson, Palfrey and Srivastava (1994) researched implementation in undominated Nash equilibria, using bounded mechanisms. But, in all these mechanisms, each agent’s strategy space is very large: he announces at least his own preference relation, another agent’s preference relation, an outcome and an integer. It is not easy to apply these mechanisms in real societies.

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Several papers researched implementation in specific environments to design reasonable mechanisms implementing given SCCs. Hurwicz (1979) constructed smooth mechanisms implementing the Walrasian correspondence. His mechanism satisfies balancedness requiring that total demand be equal to total supply for any strategy profile. Schmeidler (1980) constructed a balanced mechanism which implements the Walrasian correspondence, where each agent announces a price vector and his own consumption bundle, but not preference relations. Since their mechanisms implement specific SCCs, we do not know whether these mechanisms implement other SCCs or not. In addition, the economic meaning of strategy spaces in their mechanisms is not clear.

To make strategy spaces of a mechanism meaningful, Dutta, Sen and Vohra (1995) constructed reasonable mechanisms, “elementary” mechanisms, in economic environments. Saijo, Tatamitani and Yamato (1996) constructed “natural” mechanisms in private goods economies. Also, to characterize SCCs to be implemented by their mechanisms, Dutta, Sen and Vohra (1995) and Saijo, Tatamitani and Yamato (1996) found conditions for SCCs to be implemented by each of these reasonable mechanisms in economic environments.

In this paper, we apply natural mechanisms by Saijo, Tatamitani and Yamato to divisible pure public goods economies, and consider conditions for SCCs to be implemented in Nash equilibria by these natural mechanisms. A pure public good is one for which there are non-rivalry and non-excludability in consumption. Non-rivalry means that if the good is consumed by agent $i$, this does not preclude agent $j$ from consuming it. Non-excludability means that agent $i$ can consume the good even if he does not pay for it. In public goods economies, there exist free-rider problems because of the properties of public goods. Also, prices of public goods should be different across individuals to achieve Pareto efficiency. So, we consider four types of “natural” mechanisms in which each agent announces (i) his own consumption bundle of private goods and an input vector for public goods, (ii) a price vector of private goods, his own personalized price vector of public goods, his own consumption bundle of private goods and an input vector for public goods, (iii) a price vector of private goods, his neighbor’s personalized price vector of public goods, his own consumption bundle of private goods and an input vector for public goods, and (iv) a price vector of private goods, all agents’ personalized price vectors of public goods, his own consumption bundle of private goods and an input vector for public goods. We call these mechanisms quantity (Q), individual price-quantity (IPQ), price\textsuperscript{2}-quantity (P\textsuperscript{2}Q), and price\textsuperscript{n}-quantity (P\textsuperscript{n}Q) mechanisms respectively. We need information about an input vector for public goods which is important for finding “potential deviators,” because a production function of public goods may not have an inverse function. Q and IPQ mechanisms may be more self-relevant than P\textsuperscript{2}Q and P\textsuperscript{n}Q mechanisms in the sense that in Q and IPQ mechanisms, each agent announces only his own information.

In addition to making strategy spaces economically meaningful, we require forthrightness, individual feasibility, balancedness, and the best response property as desirable conditions for mechanisms. Forthrightness, introduced by Saijo, Tatamitani and Yamato (1996), requires that each agent receive his own consumption which he has announced in equilibria. Individual feasibility means that, for each agent, a mechanism assigns a consumption bundle in his consumption set as an outcome. Balancedness requires that total demand be equal to total supply for any strategy profile. The best response property introduced by Jackson, Palfrey and Srivastava (1994) requires that for any strategy combination of the other agents, each agent have a best response. We call mechanisms satisfying forthrightness, individual feasibility, balancedness, and the best response property “natural” mechanisms.

Our results are as follows. First, regardless of the number of goods, the class of Pareto efficient SCCs implemented by natural price\textsuperscript{2}-quantity mechanisms is equivalent to that of Pareto efficient SCCs implemented by natural price\textsuperscript{n}-quantity mechanisms. Second, in the case where there exists only one public good, natural Q, IPQ, P\textsuperscript{2}Q and P\textsuperscript{n}Q mechanisms implement the Lindahl correspondence for any number of private goods. In the case where there exist more than one public goods, natural P\textsuperscript{2}Q mechanisms and natural P\textsuperscript{n}Q mechanisms implement the Lindahl correspondence but neither natural Q mechanism nor natural IPQ mechanism implements the Lindahl correspondence. Third, the Pareto correspondence is not implemented by Q, IPQ, P\textsuperscript{2}Q or P\textsuperscript{n}Q mechanisms, regardless of the number of goods. To implement the Pareto correspondence, we must design new natural mech-
anisms: price\(^2\)-quantity\(^2\) (P\(^2\)Q\(^2\)), price\(^2\)-allocation (P\(^2\)A), price\(^n\)-quantity\(^2\) (P\(^n\)Q\(^2\)), and price\(^n\)-allocation (P\(^n\)A) mechanisms.

Many papers researched implementation in public goods economies. Groves and Ledyard (1977) researched implementing Pareto efficient SCCs in public goods economies. Hurwicz (1979), Walker (1981) and Tian and Li (1991) constructed mechanisms which implement the Lindahl correspondence in public goods economies. Since their mechanisms implement specific SCCs, we do not know whether these mechanisms implement other SCCs or not. Also, the economic meaning of strategy spaces in their mechanisms is not clear. Tian and Li (1991) constructed single-valued, individually feasible, balanced, and continuous mechanisms which implement Lindahl correspondence in divisible public goods. But strategy space of his mechanisms is very large.

Saijo, Tatamitani and Yamato (1996) constructed natural mechanisms which satisfy forthrightness, individual feasibility, balancedness and the best response property in pure exchange economies without public goods. Since their mechanisms satisfy forthrightness, there exists no information smuggling problem. Saijo, Tatamitani and Yamato considered six types of natural mechanisms: quantity, quantity\(^2\), allocation, price-quantity, price-quantity\(^2\), and price-allocation mechanisms. Saijo, Tatamitani and Yamato found conditions for SCCs to be implemented in Nash equilibria by each of these natural mechanisms and showed that natural price-quantity mechanisms implement the fair correspondence and the Walrasian correspondence and price-quantity\(^2\) (equivalent to price-allocation) mechanisms implement the Pareto correspondence, too. Tatamitani (2001) considered self-relevant mechanisms and characterized SCCs to be implemented by self-relevant mechanisms in general environments. Furthermore Tatamitani (2002) applied self-relevant mechanisms to pure exchange economies with private goods and researched the relation among SCCs implemented by self-relevant mechanisms, ones implemented by mechanisms which Dutta, Sen and Vohra constructed, and ones implemented by mechanisms which Saijo, Tatamitani and Yamato constructed. Yoshihara (1999) and Yoshihara (2000) designed natural mechanisms in production economies. But their researches are done in only private goods economies.

This paper is organized as follows. In Section 2, we set our basic model. In Section 3, we assume that there exist one private good and one divisible pure public good in economic environments and introduce four types of mechanisms: quantity, individual price-quantity, price\(^2\)-quantity, price\(^n\)-quantity mechanisms. Then we characterize SCCs to be implemented in Nash equilibria by each of these four “natural” mechanisms in public goods economies. It is shown that the Lindahl correspondence is implemented by Q, IPQ, P\(^2\)Q and P\(^n\)Q mechanisms in economies with one private good and one public good. In Section 4, we assume that there exist more than one private goods and one divisible pure public good in economic environments and following Section 3, we characterize SCCs to be implemented in Nash equilibria by each of these four “natural” mechanisms. It is shown that in economies with one public good, the Lindahl correspondence is implemented by four natural mechanisms for any number of private goods. In Section 5, we assume that there exist more than one private goods and more than one divisible pure public goods in economic environments. It is shown that in economies with more than one public goods, the Lindahl correspondence is not implemented by neither individual price-quantity mechanism nor quantity mechanism for any number of private goods. Section 6 contains concluding remarks and open problems. In Appendix, we show some proofs.

2 The Basic Model

There are \(l\) divisible private goods and \(m\) divisible pure public goods, where \(l \geq 1\) and \(m \geq 1\). Let \(I = \{1, \ldots, n\}\) be the finite set of agents, with generic element \(i\). We assume \(n \geq 3\). Agent \(i\)'s initial endowment (private goods) is \(\omega_i \in \mathbb{R}_+^l \setminus \{0\}\) and the aggregate endowment is denoted by \(\Omega = \sum_{i \in I} \omega_i \in \mathbb{R}_+^l\). Agent \(i\)'s preference relation is represented as a utility function \(u_i : \mathbb{R}_+^{l+m} \to \mathbb{R}\). Let \((u_1, \ldots, u_n) = u \in U = U_1 \times \cdots \times U_n\) where \(U_i\) is the class of
utility functions admissible for agent $i$. An economy is specified by $u$. Consumer $i$’s consumption of private goods is $x_i \in \mathbb{R}^l_i$ and the level of public goods is $y \in \mathbb{R}^n$. Let $x = (x_1, \ldots, x_n)$, $x_i = (x_{i1}, \ldots, x_{il})$ and $y = (y_1, \ldots, y_m)$.

$T(v, y) = 0$ means the technology of our economy, where $v \in \mathbb{R}^m_+$ represents the input vector of private goods $(v = (v^1, \ldots, v^m))$ and $y \in \mathbb{R}^n$ is the vector of outputs of public goods. We assume that $T$ is continuously differentiable. Then, the Lindahl-Bowen-Samuelson (LBS) condition is that $\sum_{i\in I} \frac{\partial u_i}{\partial y} = -\frac{\partial T}{\partial y}$ for all $k = 1, \ldots, m$ and all $j = 1, \ldots, l$.

We assume that there are constant returns to scale. This implies that the maximum profit of the production sector is zero in the economies with constant returns to scale. This is the familiar fact and agent $i$’s budget is only $p \cdot \omega_i$ at the Lindahl equilibrium if there are constant returns to scale. Then, let $y = f(v)$ and $T(v, f(v)) = 0$. We assume that $f(v) = (f_1(v^1), \ldots, f_m(v^m)) = (y_1, \ldots, y_m)$, where $f : \mathbb{R}^m_+ \rightarrow \mathbb{R}^m_+$ is a function which is concave, differentiable, homogeneous of degree one, and strictly increasing in $v$. This implies that one production sector produces one public good from private goods separately. We fix $f$ and all agents know $f$.

Moreover, let $Q_{x_i} = \{x_i \in \mathbb{R}^l_i | x_i \leq \Omega\}$, $Q_v = \{v \in \mathbb{R}^m_+ | \sum_{i=1}^m v^k \leq \Omega\}$, $Q_y = \{y \in \mathbb{R}^n_+ | \sum_{k=1}^m v^k \leq \Omega$ and $f(v) = y\}$, $Q = \{(x, v) \in \mathbb{R}^{l+m}_+ | x + \sum_{i=1}^m v^k \leq \Omega\}$, and $Q_i = \{(x, y) \in \mathbb{R}^{l+m}_+ | (x, v) \in Q$ and $f(v) = y\}$. The set of feasible outcomes is denoted by $A$. In public goods economies, the set of feasible outcomes is $A = \{(x, y) \in \mathbb{R}^{l+m}_+ | \sum_{i=1}^m x_i + \sum_{i=1}^m v^k = \Omega$ and $f(v) = y\}$. Let $\Lambda = \{(x, y) \in \mathbb{R}^{l+m}_+ | (x, y) \in A\}$ be the set of interior feasible outcomes in public goods economies. Free disposal is disallowed.

A social choice correspondence (an SCC) is a correspondence $F : U \rightarrow A$.

Agent $i$’s strategy is denoted by $s_i$ and agent $i$’s strategy space is $S_i$, where $(s_1, \ldots, s_n) = s \in S = S_1 \times \cdots \times S_n$. A mechanism is a pair $\Gamma = (S, g)$ where $g : S \rightarrow A$ is the outcome function. Denote the $i$-th component of $g(s)$ by $g_i(s)$. Given $s \in S$, let $g^x(s)$ be the bundles of private goods assigned by $g$, $g^y(s)$ be the bundles of public goods assigned by $g$ and $g(s) = (g^x(s), g^y(s)) \in \mathbb{R}^{l+m}_+$, where $g^x(s) = (g^x_1(s), \ldots, g^x_n(s))$ and $g^y(s) \in \mathbb{R}^l_i$ for all $i \in I$.

Let $s_{-i}$ be $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ and $g_i(s_i, s_{-i}) = \{a_i \in Q_i | a_i = g(s'_i, s_{-i})$ for some $s'_i \in S_i\}$ be the attainable set of participant $i$ at $s_{-i}$.

For $i \in I$, $u_i \in U_i$, and $(x, y) \in \mathbb{R}^l_i \times \mathbb{R}^n_+$, let $L((x, y), u_i) = \{(x'_i, y') \in Q_i | u_i(x_i, y) \geq u_i(x'_i, y')\}$ be the weak lower contour set for agent $i$ with $u_i$ at $(x, y)$.

Given $u$, let $N_g(u)$ be the set of Nash equilibria of $\Gamma = (S, g)$, i.e., $N_g(u) = \{s \in S | \forall i \in I, u_i(g_i(s)) \geq u_i(g_i(s'_i, s_{-i}))$ for all $s'_i \in S_i\} = \{s \in S | \forall i \in I, g_i(s_i, s_{-i}) \subseteq L(g_i(s), u_i)\}$. Let $N_g : U \rightarrow S$ be a Nash equilibrium correspondence, $g \circ N_g(u)$ be the set of Nash equilibrium outcomes of $\Gamma = (S, g)$, i.e., $g \circ N_g(u) = \{a \in A | \exists s \in N_g(u)$ s.t. $g(s) = a\}$.

Definition (Implementation): We say that the mechanism $\Gamma = (S, g)$ implements an SCC $F$ if for all $u \in U$, the set of equilibrium outcomes of the $\Gamma = (S, g)$ coincides with the set represented as $F(u)$, i.e., $F(u) = g \circ N_g(u)$.

If there exists a mechanism that implements an SCC $F$, the SCC is said to be implementable.

Definition (Monotonicity (Maskin (1977))): For all $u, u' \in U$ and all $(x, y) \in F(u)$, if for all $i \in I$ and all $(x', y') \in A$, $u_i(x_i, y) \geq u_i(x'_i, y')$ implies $u'_i(x_i, y) \geq u'_i(x'_i, y')$, i.e., $L((x, y), u_i) \subseteq L((x', y), u'_i)$ for all $i \in I$, then $(x, y) \in F(u')$.

Maskin (1977) showed that the monotonicity is necessary for SCCs to be implementable in Nash equilibrium.

We introduce two monotonic social choice correspondences.

(Weak) Pareto Correspondence: $P(u) = \{(x, y) \in A | (x', y') \in A$ s.t. $u_i(x'_i, y') > u_i(x_i, y) \forall i \in I\} = \{(x, y) \in A | \forall (x'_i, y') \in A \exists i \in I$ s.t. $u_i(x_i, y) \geq u_i(x'_i, y')\}$

Notice that in economies with $u$ which is continuous and strictly monotone, the weak Pareto correspondence is
equivalent to the strong Pareto correspondence.

Let \( p = (p_1, \ldots, p_l) \in \mathbb{R}_+^l \) be a price vector of private goods, \( q_i = (q_{i1}, \ldots, q_{in}) \in \mathbb{R}_+^m \) be agent \( i \)'s personalized price vector of public goods, and \( q = (q_1, \ldots, q_n) \in \mathbb{R}_+^{nm} \) be personalized price vectors of public goods.

**Lindahl Correspondence:** \( L(u) = \{(x, y) \in A| y \leq f(v) \text{ and } \exists(p, q) \in \mathbb{R}_+^{l+nm} \text{ s.t. } \forall i \in I, p \cdot x_i + q_i \cdot y \leq p \cdot \omega_i \text{ and } \forall(x'_i, y'_i) \in \mathbb{R}_+^{l+m}, [p \cdot x'_i + q_i \cdot y'_i \leq p \cdot \omega_i \Rightarrow u_i(x_i, y) \geq u_i(x'_i, y'_i)] \text{ and } \sum_{i \in I} q_i \cdot y - p \cdot \sum_{k=1}^m v^k \geq \sum_{i \in I} q_i \cdot y' - p \cdot \sum_{k=1}^m v^k \forall(v', y') \in \mathbb{R}_+^{l+m} \text{ s.t. } y' \leq f(v') \}\)

We assume the following notions.

**Definition (E):** For all \( u \in U, F(u) \subseteq P(u); \text{ efficiency} \)

**Definition (I):** For all \( u \in U, F(u) \subseteq \hat{A}; \text{ interiority} \)

**Assumption (C):** \( U \) is the set of profiles of utility functions \( u \) such that for all \( i, u_i \) is a continuously differentiable, quasi-concave and strictly monotonic function.

**Assumption (I):** For all \( u \in U, F(u) \subseteq \hat{A}. \)

**Assumption (D):** For all \( u \in U \) and \( i \in I, \) if \( (x, y) \in A \) is such that \( x_i = 0 \) for all \( i, \) then there exists \( (x', y') \in A \) such that \( u_i(x'_i, y') > u_i(x_i, y) \) for all \( i \in I. \)

In this paper, let these above assumptions be satisfied. By Assumption (I), the Lindahl correspondence satisfies Monotonicity. Assumption (D) cannot be dispensed with the proof of sufficiency for Nash implementability. This assumption implies that it is undesirable for all agents that the entire endowment is transformed into the public goods.

3 Economies with One Private Good and One Public Good

We apply quantity mechanisms and price-quantity mechanisms which Saijo et al. (1996) constructed in private goods economies to public goods economies. For notational ease, we consider the case where one private good and one divisible pure public good exist. More generalized cases are stated in the later section. We assume the following situation.

Let \( l = 1, m = 1 \) and \((p, q) = (p_1, q_1, \ldots, q_n) \) where \( p_i, q_i \in \mathbb{R}_+ \) for all \( i \in I. \) \( Du_i(x_i, y) = \frac{\partial u_i}{\partial y} / \frac{\partial y}{\partial x_i} \in [0, 1] \) is agent \( i \)'s marginal rate of substitution of \( x_i \) for \( y \) at \( (x_i, y). \) Let \( T(v, y) = T(v, f(v)) = 0, \) i.e., \( y = f(v), \) where \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) which is concave, differentiable, homogeneous of degree one, and strictly increasing in \( v. \) Constant returns to scale imply that for all \( \lambda \in \mathbb{R}, f(\lambda v) = \lambda f(v) = \lambda y. \) Let \( y = f(v) = v. \) Notice that in this model, inputs for a public good are equal to outputs of the public good. Since \( f(\lambda v) = \lambda v = \lambda y, \) there are constant returns to scale in this economy. Also, this implies that \( T(v, y) = v - y = 0 \) and that the marginal rate of transformation (MRT) of \( v \) for \( y \) at \((v, y)\) is \(-\frac{\partial T}{\partial y} / \frac{\partial T}{\partial v} = 1. \) A social planner collects the costs for a public good proportionally to each agent’s personalized price and supplies the public good, maximizing the profit. We assume that there are constant returns to scale but do not consider the production sector explicitly to let agent \( i \)'s budget be always only \( p_i q_i. \) When \( \sum_{i \in I} Du_i(x_i, y) = 1, \) the Lindahl-Bowen-Samuelson condition is satisfied at \((x, y). \) Let \( p_1 = 1. \) Then, \( \frac{v}{p_i} = q_i \) for all \( i \in I. \) If \((x, y)\) is the Lindahl equilibrium allocation with the equilibrium prices \((p, q), \) then
\[Du_i(x_i, y) = \frac{q_i}{p_i} = q_i \text{ for all } i \in I. \] Moreover, we assume that there is no transfer of tax, i.e., \( q_i \in [0, 1] \) for all \( i \in I \).

3.1 Natural Quantity Mechanisms

We define terms, before we define the mechanisms. The mechanism \( \Gamma = (S, g) \) is \textit{individually feasible} if \( g(s) \in \mathbb{R}^{n+1} \) for all \( s \in S \). The mechanism \( \Gamma = (S, g) \) is \textit{balanced} if \( \sum_{i \in I} (\partial g_i^x(s) + \partial g_i^y(s)) = \Omega \) for all \( s \in S \).

**Definition (The Best Response Property (Jackson, Palfrey, and Srivastava (1994)))**: The mechanism \( \Gamma = (S, g) \) satisfies the best response property if for all \( i \in I \), all \( u_i \in U_i \), and all \( s_{-i} \in S_{-i} \), there exists \( s_i \in S_i \) such that \( u_i(g_i(s_i, s_{-i})) \geq u_i(g_i(s'_i, s_{-i})) \) for all \( s'_i \in S_i \).

Saijo et al. (1996) defined a natural quantity mechanism in private goods economies. We directly rewrite this for public goods economies.

**Definition (Natural Quantity (Q) Mechanisms)**: The SCC \( F \) is implementable by a natural quantity mechanism if there exists a mechanism \( \Gamma = (S, g) \) such that

(i) \( \Gamma \) implements \( F \);

(ii) for all \( i \in I \), \( S_i = Q \);

(iii) for all \( u \in U \) and all \((x, y) \in F(u)\), if \( s_i = (x_i, y) \) for all \( i \in I \), then \( s \in N(u) \) and \( g(s) = (x, y) \);

(iv) \( \Gamma \) is individually feasible and balanced; and

(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces only his own consumption bundle of the private good and an input vector for the public good. However, an input vector for a public good is equivalent to the level of the public good since we assume that \( y = v \) in this section. Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy. We will find necessary and sufficient conditions for an SCC to be Nash implementable by Q mechanisms. Following Saijo et al. (1996), let \( F^{-1}(x, y) \equiv \{u' \in U | (x, y) \in F(u')\} \) and \( \Lambda^F(x, y) \equiv \cap_{u' \in F^{-1}(x, y)} L((x_i, y), u'_i) \). To public goods economies, we apply Condition \( W^* \) which Saijo et al. (1996) introduced.

**Definition (Condition \( W^* \))**: For all \( u, u' \in U \) and all \((x, y) \in F(u)\), if \( \Lambda^F(x, y) \subseteq L((x_i, y), u'_i) \) for all \( i \in I \), then \((x, y) \in F(u') \).

We need another condition for sufficiency that Pareto efficient SCCs are Nash implementable by Q mechanisms. Since each agent announces only his own quantity except a price vector, the punishment condition is needed to prevent agents from lying. Let \( I(x, y) \equiv \{i \in I | F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) \neq \emptyset \} \).

**Definition (Condition \( Q \))**: For every \((x, y) \in Q_{x_1} \times \cdots \times Q_{x_n} \times Q_y \) such that \( I(x, y) = I \), there exists \( z(x, y) \in A \) such that

(i) \( z_i(x, y) \in \Lambda^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) \) for all \( i \in I \); and

(ii) if there exists \( u^* \in U \) such that \( \Lambda^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) \subseteq L(z_i(x, y), u^*_i) \) for all \( i \in I \), then \( z(x, y) \in F(u^*) \).

This is the condition for punishment. In public goods economies, this condition corresponds to Condition \( PQ \).
in private goods economies. We obtain the following result.

**Proposition 3.1:** Let \( n \geq 3, \ l = 1 \) and \( m = 1 \). An SCC is implementable by a natural quantity mechanism if and only if it satisfies Condition \( W^* \) and Condition \( Q \).

The proof of Proposition 3.1 is in the appendix. To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \).

This result is parallel to the result concerning Q mechanisms in private goods economies by Saijo et al. (1996).

### 3.2 Natural Individual Price-Quantity Mechanisms

Saijo et al. (1996) defined a natural price-quantity mechanism in private goods economies. For \( x \in \mathbb{R}^{nl} \) and \( u \in U \), let \( \Pi(x, u) = \{ p \in \Delta \mid p = Du_i(x_i) \text{ for all } i \in I \} \), where \( \Delta \subseteq \mathbb{R}_+^l \) is the unit simplex.

**Definition (Natural Price-Quantity Mechanisms (Saijo et al. (1996)))**: The SCC \( F \) is implementable by a natural price-quantity mechanism if there exists a mechanism \( \Gamma = (S, g) \) such that

(i) \( \Gamma \) implements \( F \);

(ii) for all \( i \in I \), \( S_i = \Delta \times Q \);

(iii) for all \( u \in U \) and all \( x \in F(u) \), if \( \Pi(x, u) = \{ p \} \) and \( s_i = (p, x_i) \) for all \( i \in I \), then \( s \in N_g(u) \) and \( g(s) = x \);

(iv) \( \Gamma \) is individually feasible and balanced; and

(v) \( \Gamma \) satisfies the best response property.

We directly rewrite this for public goods economies. From here, let the unit simplex \( \Delta \subseteq \mathbb{R}_+^n \). Since the marginal rate of transformation of \( v \) for \( y \) at \( (v, y) \) is \( \frac{\partial T}{\partial y} \frac{\partial y}{\partial v} = 1 \) and a price of the private good \( p = 1 \), a personalized price vector \( q = (q_1, \ldots, q_n) \in \Delta \) if the LBS condition is satisfied.

**Definition (Natural Individual Price-Quantity (IPQ) Mechanisms)**: The SCC \( F \) is implementable by a natural individual price-quantity mechanism if there exists a mechanism \( \Gamma = (S, g) \) such that

(i) \( \Gamma \) implements \( F \);

(ii) for all \( i \in I \), \( S_i = [0, 1] \times Q \);

(iii) for all \( u \in U \) and all \( (x, y) \in F(u) \), if \( q_i = Du_i(x_i, y) \) for all \( i \in I \) and \( s_i = (q_i, (x_i, y)) \) for all \( i \in I \), then \( s \in N_g(u) \) and \( g(s) = (x, y) \);

(iv) \( \Gamma \) is individually feasible and balanced; and

(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces a price vector of the private good, his own personalized price vector of the public good, his own consumption bundle of the private good and an input vector for the public good (equivalent to the level of the public good). Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy.

This mechanism may be more self-relevant than \( P^2Q \) (in subsection 3.3) and \( P^nQ \) mechanisms (in subsection 3.4) in the sense that in this mechanism, each agent announces only his own information, i.e., his personalized price vector and his own quantity.

Then, since there exist free-rider problems in public goods economies, a social planner can not identify the deviator from a desirable outcome. But, by constructing the coordination game with \( y \) in the proof of sufficiency,
Proposition 3.2: Let all (i) \( \Gamma \) implements satisfies Condition Then a Pareto efficient SCC is implementable by a natural individual price-quantity mechanism if and only if it

Definition (Condition \( M \)): For all \( u, u' \in U \) and all \((x, y) \in F(u)\), if \( \Lambda^F((x, y), u) \subseteq L((x_i, y), u'_i) \) for all \( i \in I \), then \((x, y) \in F(u')\).

We need another condition for sufficiency that Pareto efficient SCCs are Nash implementable by IPQ mechanisms. Since each agent announces only his own quantity except a price vector, the punishment condition is needed to prevent agents from lying. We define \( F^{-1}(x, y, y) \equiv \{u' \in U:\ (x, y) \in F(u')\, q_i = Du_i(x, y) \) for all \( i \in I \) and \( q \in \Delta\), \( \Lambda^F((x, y), q) \equiv \cap_{u' \in F^{-1}(x, y, q)} L((x_i, y), u'_i) \), and \( I^*(q, (x, y)) \equiv \{i \in I\ \cap_{u' \in F^{-1}((\Omega-y-\sum_{j \neq i} x_j), x_{-i}), (1-\sum_{j \neq i} q_j, q_{-i})) = \emptyset\} \) for all \( i \in I \), under \( u' \in F^{-1}((\Omega-y-\sum_{j \neq i} x_j), x_{-i}), (1-\sum_{j \neq i} q_j, q_{-i})) \), a social planner thinks that agent \( i \) should announce the remainder about his consumption vector and a price vector.

When we construct the mechanism to implement a Pareto efficient SCC \( F \), the Lindahl-Bowen-Samuelson condition must be satisfied at an equilibrium. In this section, the LBS condition is that \( \sum_{i \in I} q_i = 1 \). Suppose that each agent \( i \) announces \((q_i, (x_i, y))\) such that \((x, y) \in A \) and \( \sum_{i \in I} q_i \neq 1 \). If the social planner can find a preference profile \( u \in U \) such that \((x, y) \in F(u)\) and \( q_i = Du_i(x_j, y) \) for all \( j \neq i \), he concludes that agent \( i \) should be punished. We call such an agent a potential deviator, since agent \( i \) should have announced \( 1-\sum_{j \neq i} q_j \) rather than \( q_i \). If a social planner can not find such a preference profile, agents other than agent \( i \) might have announced wrong prices. Therefore, agent \( i \) is not counted as a potential deviator. But, when all agents are potential deviators, we can not assign them zero allocation as punishment since we demand feasibility. Then, we define the following condition.

Definition (Condition \( IPQ \)): For every \((q, (x, y)) \in [0, 1]^n \times Q_x \times \cdots \times Q_x \times Q_y\) such that \( I^*(q, (x, y)) = I \), there exists \( z(q, (x, y)) \in A \) such that

(i) \( z(q, (x, y)) \in \Lambda^F(((\Omega-y-\sum_{j \neq i} x_j), x_{-i}), (1-\sum_{j \neq i} q_j, q_{-i})) \) for all \( i \in I \); and

(ii) if there exists \( u' \in U \) such that \( \Lambda^F(((\Omega-y-\sum_{j \neq i} x_j), x_{-i}), y), (1-\sum_{j \neq i} q_j, q_{-i})) \subseteq L(z(q, (x, y)), u'_i) \) for all \( i \in I \), then \( z(q, (x, y)) \in F(u')\).

This condition is a variant Condition \( PQ \) which Saijo et al. (1996) introduced.

Proposition 3.2: Let \( n \geq 3 \), \( l = 1 \) and \( m = 1 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i, u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural individual price-quantity mechanism if and only if it satisfies Condition \( M \) and Condition \( IPQ \).

The proof of Proposition 3.2 is in the appendix. To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \).

3.3 Natural Price\(^2\)-Quantity Mechanisms

Definition (Natural Price\(^2\)-Quantity (P\(^2\)Q) Mechanisms): The SCC \( F \) is implementable by a natural price\(^2\)-quantity mechanism if there exists a mechanism \( \Gamma = (S, g) \) such that

(i) \( \Gamma \) implements \( F \);
(ii) for all \( i \in I, S_i = [0, 1]^2 \times Q \);
(iii) for all \( u \in U \) and all \((x, y) \in F(u)\), if \( q_i = Du_i(x_i, y) \) for all \( i \in I \) and \( s_i = (q_i, q_{i+1}, (x_i, y)) \) for all \( i \in I \), then \( s \in N_g(u) \) and \( g(s) = (x, y) \) where \( n + 1 = 1 \);
(iv) \( \Gamma \) is individually feasible and balanced; and
(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces a price vector of the private good, his own personalized price vector of the public good, his neighbor’s personalized price vector of the public good, his own consumption bundle of the private good and an input vector for the public good (equivalent to the level of the public good). Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy. We will find necessary and sufficient conditions for an SCC to be Nash implementable by P\textsuperscript{2}Q mechanisms.

We need a new condition for sufficiency that Pareto efficient SCCs are Nash implementable by P\textsuperscript{2}Q mechanisms. Since each agent announces only his own quantity except a price vector, the punishment condition is needed to prevent agents from lying. We define \( I(q, (x, y)) = \{ i \in I | F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \neq \emptyset \} \).

**Definition (Condition \( P^nQ \))**: For every \((q, (x, y)) \in \Delta \times Q_{x_1} \times \cdots \times Q_{x_n} \times Q_y \) such that \( I(q, (x, y)) = I \), there exists \( z(q, (x, y)) \in A \) such that
(i) \( z_i(q, (x, y)) \in \Lambda^P_i(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \) for all \( i \in I \); and
(ii) if there exists \( u^* \in U \) such that \( \Lambda^P_i(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \subseteq L(z_i(q, (x, y), u^*_i) \) for all \( i \in I \), then \( z(q, (x, y)) \in F(u^*) \).

This is the condition for punishment. In public goods economies, this condition corresponds to Condition \( PQ \) in private goods economies. In P\textsuperscript{2}Q mechanisms, since each agent announces another agent’s information, we can use P\textsuperscript{2}Q-Consistency defined as follows. Then, a condition for punishment, i.e., Condition \( P^nQ \), is weaker than Condition \( IPQ \).

When we construct the mechanism for the proof of sufficiency, we need the following notion.

**Definition (P\textsuperscript{2}Q-Consistency)**: Let \( s_i = (q_i^j, q_{i+1}^j, (x_i^j, y_i^j)) \in S_i = [0, 1]^2 \times Q_i = \Delta \subseteq \mathbb{R}_{n+1}^2 \), and \((x, y) \in A \) be given. We say that \( s \) is P\textsuperscript{2}Q-consistent with \( q \) and \( y \) if for all \( j \in I \), \( q_j^i = q_j^{i+1} = q_j \) and \( y_j^i = y_j \).

Given \( i \in I \), the strategy profile \( s_{-i} \in S_{-i} \) is P\textsuperscript{2}Q-consistent with \( q \) and \( y \) if for all \( j \neq i \), \( i + 1 \), \( q_j^i = q_j^{i+1} = q_j \), \( q_{i+1}^i = q_{i+1} \) and for all \( j \neq i \), \( y_j = y \).

This notion is a variant of Consistency introduced by Saijo et al. (1996). When \( n \geq 3 \), in P\textsuperscript{2}Q mechanisms, a social planner can identify the agent deviating from an equilibrium, using the above notion. To see this, let the true personalized price vector be \( q \), under the true preference profile \( u \in U \) and \((x, y) \in F(u) \). In a P\textsuperscript{2}Q mechanism, the strategy profile \( s \) where \( s_i = (q_i, q_{i+1}, (x_i, y)) \) for all \( i \in I \) constitutes an equilibrium at \( u \) by forthrightness. Suppose that agent \( d \) deviates from \((q_d, q_{d+1}) \) to \((q_d', q_{d+1}') \), while the other agents do not change their announcements. There are three possible cases. In case 1 where \( q_d \neq q_d' \) and \( q_{d+1} \neq q_{d+1}' \), clearly, a social planner can identify agent \( d \) as a unique deviator. In case 2 where \( q_d \neq q_d' \) and \( q_{d+1} = q_{d+1}' \), since \((q_d', q_{d-1}) \) is not feasible by the Lindahl-Bowen-Samuelson condition, a social planner can identify agent \( d \) as a unique deviator.

The case 3 where \( q_d = q_d' \) and \( q_{d+1} \neq q_{d+1}' \) is analogous with the case 2.

When \( n = 2 \), we can not well define the term that \( s_{-i} \in S_{-i} \) is P\textsuperscript{2}Q-consistent with \( q \) and \( y \) since a social planner can not know the deviator \( i \) in the definition. Using the notion, P\textsuperscript{2}Q-Consistency, we reduce the strategy space in the mechanism which implements an SCC satisfying Condition \( M \) and Condition \( P^nQ \).
Proposition 3.3: Let \( n \geq 3, \ l = 1 \) and \( m = 1 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i, u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural price\(^2\)-quantity mechanism if and only if it satisfies Condition \( M \) and Condition \( P^mQ \).

The proof of Proposition 3.3 is in the appendix. To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \). This result is parallel to the result concerning PQ\(^2\) mechanisms in private goods economies by Saijo et al. (1996).

3.4 Natural Price\(^n\)-Quantity Mechanisms

We show why the mechanisms need not order each agent to announce every personalized price to implement Pareto efficient SCCs satisfying Condition \( M \) and Condition \( P^mQ \).

Definition (Natural Price\(^n\)-Quantity (P\(^n\)Q) Mechanisms): The SCC \( F \) is implementable by a natural price\(^n\)-quantity mechanism if there exists a mechanism \( \Gamma = (S, g) \) such that

(i) \( \Gamma \) implements \( F \);
(ii) for all \( i \in I, S_i = \Delta \times Q \);
(iii) for all \( u \in U \) and all \( (x, y) \in F(u) \), if \( q_i = Du_i(x_i, y) \) for all \( i \in I \) and \( s_i = (q, (x_i, y)) \) for all \( i \in I \), then \( s \in N_g(u) \) and \( g(s) = (x, y) \);
(iv) \( \Gamma \) is individually feasible and balanced; and
(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces a price vector of the private good, all agents’ personalized price vectors of the public good, his own consumption bundle of the private good and an input vector for the public good (equivalent to the level of the public good). Condition (ii) requires that each agent’s strategy satisfy the LBS condition. Condition (iii) is forthrightness that is introduced by Saijo et al. (1996). We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy. We will find necessary and sufficient conditions for an SCC to be Nash implementable by P\(^n\)Q mechanisms. But, we do not need new conditions. We obtain the following result.

Proposition 3.4: Let \( n \geq 3, \ l = 1 \) and \( m = 1 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i, u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural price\(^n\)-quantity mechanism if and only if it satisfies Condition \( M \) and Condition \( P^mQ \).

The proof of Proposition 3.4 is in the appendix. Each agent need not announce other agents’ consumption bundles, since desirable quantities of public goods are not provided for each agent unless all announcements about quantity of public goods coincide. To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \).

By Proposition 3.3 and Proposition 3.4, clearly, we obtain the following result.

Proposition 3.5: Let \( n \geq 3, \ l = 1 \) and \( m = 1 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i, u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural price\(^n\)-quantity mechanism if and only if it is implementable by a natural price\(^2\)-quantity mechanism.

This result is parallel to the result concerning PA mechanisms in private goods economies by Saijo et al. (1996). To see the equivalence between a P\(^n\)Q mechanism and a P\(^2\)Q mechanism, let \( F(u) \subseteq P(u) \) for all \( u \in U \) and the
true personalized price vector be $q_i$ under the true preference profile $u \in U$ and $(x, y) \in F(u)$. In a P⁰Q mechanism, the strategy profile $s$ where $s_i = (q_i, (x_i, y_i))$ for all $i \in I$ constitutes an equilibrium at $u$ by forthrightness. In a P²Q mechanism, the strategy profile $s$ where $s_i = (q_i, (x_i, y_i))$ for all $i \in I$ constitutes an equilibrium at $u$ by forthrightness. Since $F(u) \subseteq P(u)$ for all $u \in U$, $q \in \Delta$ and $\sum_{i \in I} q_i = 1$ by the LBS condition. Let the number of deviators from $s$ be $r$. Without loss of generality, let agent 1, agent 2, ..., agent $r$ be deviators. Suppose that agent 1, ..., agent $r$ deviate from $q$ to $q^1, \ldots, q^r$ respectively, while the other agents do not change their announcements, in a P²Q mechanism and from $(q_1, q_{i+1})$ to $(q^1, q^r_{i+1})$ respectively, while the other agents do not change their announcements in a P²Q mechanism. We will see that such deviation does not occur in either P²Q mechanism or P⁰Q mechanism.

In a P⁰Q mechanism, it is clear that $(s_1, \ldots, s_r, \ldots, s_n)$ is not desirable because a social planner knows $q \neq q^d$ for each $d = 1, \ldots, r$ when $r \leq n - 1$. When $r = n$, $q$ is not a price system at a Pareto efficient allocation. In a P²Q mechanism, there are three possible cases. In case 1 where $r = 1$, a social planner can identify agent 1 as a unique deviator because $s_{-1}$ is P²Q-consistent with $q$. In case 2 where $2 \leq r \leq n - 1$, a social planner knows that $(s_1, \ldots, s_r, \ldots, s_n)$ is not desirable because $q_1 \neq q^1_1$, $q^r_{r+1} \neq q_{r+1}$ or $q^1_1 + \cdots + q^r_{r+1} + \cdots + q_n \neq 1$. If $q_1 = q^1_1, q^r_{r+1} = q_{r+1}$ and $q^1_1 + \cdots + q^r_{r+1} + \cdots + q_n = 1$, because there exist $d, d' \in \{1, \ldots, r\}$ such that $q^d_d > q_d$ and $q^d_{d'} < q_{d'}$, $(s_1, \ldots, s_r, \ldots, s_n)$ is not desirable for agent $d$ by the cost sharing rule and the assumption (C). In case 3 where $r = n$, we need not consider this case because in this case, $q$ is not desirable at $(x, y)$, that is, $q$ is not a price system at a Pareto efficient allocation. Notice that $(x, y) \in F(u) \subseteq P(u)$. Therefore, because a social planner knows that $(s_1, \ldots, s_r, \ldots, s_n)$ is not desirable in either P⁰Q mechanism or P²Q mechanism when agent 1, ..., agent $r$ deviate from $s$, a P⁰Q mechanism is equivalent to a P²Q mechanism for all Pareto efficient SCCs.

In our model, we assume complete information since we use a Nash equilibrium as an implementation solution concept. So, notice that the information level which agents must have in a P⁰Q mechanism is the same as that in a P²Q mechanism.

### 3.5 Implementability of the Lindahl Correspondence

Since each agent announces only his own information in Q mechanisms and IPQ mechanisms, there may exist free-rider problems in these mechanisms. Since each agent announces only his own information in Q mechanisms and IPQ mechanisms, there may exist free-rider problems in these mechanisms. Let $H(q_1, (x_1, y_1)) \equiv \{(x'_1, y'_1) \in \mathbb{R}_+^2 | x'_1 + q_1 y'_1 \leq x_1 + q_1 y_1\}$ be a half space and $L$ be the Lindahl correspondence. When $l = 1$ and $m = 1$, notice that $\Lambda^l_1(x, y) = \Lambda^l_1((x, y), u)$ for all $i \in I$ and all $u \in L^{-1}(x, y)$. That is, when $l = 1$ and $m = 1$, the Lindahl equilibrium price system for an allocation is uniquely determined. Then, it is clear that $H(q_1, (x_1, y_1)) \subseteq \Lambda^l_1(x, y)$ when $Du_i(x_1, y_1) = q_i$. Therefore, the following proposition is proved.

**Proposition 3.6**: Let $n \geq 3$, $l = 1$ and $m = 1$. Then, the Lindahl correspondence $L$ defined on a domain $U' \equiv \{u \in U | \omega = (\omega_1, \ldots, \omega_n) \notin L(u)\}$ satisfies Condition W* and Condition Q.

**Proof.** Take any $u \in U$ and $(x, y) \in L(u)$. Then the Lindahl equilibrium price system $q \in \Delta \subseteq \mathbb{R}_+^n$ for $(x, y) \in L(u)$ is unique since $l = 1$, $m = 1$ and $\omega \notin L(u)$ for all $u \in U'$. It is clear that $H(q_1, (\omega_1, 0)) = \{(x_1, y) \in \mathbb{R}_+^2 | \omega_1 \geq x_1 + q_1 y \}$ for all $i \in I$. If $\Lambda^l_1(x, y) \subseteq L((x_1, y), u^*_1)$ for all $i \in I$, then $(x, y) \in L(u^*)$. Therefore $L$ satisfies Condition W*.

Let $z(x, y) = \omega$ for every $(x, y) \in Qx_1 \times \cdots \times Qx_n \times Qy$ such that $I(x, y) = I$. Let $q^l \in \Delta$ be the Lindahl equilibrium price system for an allocation. When $l = 1$ and $m = 1$, notice that $\Lambda^l_1(x, y) = \Lambda^l_1((x, y), u)$ for all $i \in I$ and all $u \in L^{-1}(x, y)$. Since $q^l \in \Delta$ and it is always satisfied that $\omega_i \in \Lambda^l_1(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y)$ for all $i \in I$ by the definition of SCC $L$, it is clear that $L$ satisfies Condition Q.

This proposition implies all the four previous mechanisms, i.e., Q, IPQ, P²Q and P⁰Q mechanisms implement
the Lindahl correspondence.

**Proposition 3.7:** Natural quantity mechanisms implement the Lindahl correspondence.

## 4 Economies with / Private Goods and One Public Good

### 4.1 Natural Quantity Mechanisms

We define terms, before we define the mechanisms. The mechanism \( \Gamma = (S, g) \) is *individually feasible* if \( g(s) \in \mathbb{R}^{n_l+1}_+ \) for all \( s \in S \). The mechanism \( \Gamma = (S, g) \) is *balanced* if \( \sum_{i \in I} g_i^x(s) + v = \Omega \) and \( f(v) = g^y(s) \) for all \( s \in S \).

**Definition (Natural Quantity (Q) Mechanisms):** The SCC \( F \) is implementable by a *natural quantity mechanism* if there exists a mechanism \( \Gamma = (S, g) \) such that
(i) \( \Gamma \) implements \( F \);
(ii) for all \( i \in I, S_i = Q \);
(iii) for all \( u \in U \) and all \( (x, f(v)) \in F(u) \), if \( s_i = (x_i, v) \) for all \( i \in I \), then \( s \in N_g(u) \) and \( g(s) = (x, f(v)) \);
(iv) \( \Gamma \) is individually feasible and balanced; and
(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces his own consumption bundle of private goods and an input vector for the public good instead of the level of the public good, because \( f \) does not have the inverse function. We need information about a desirable input vector for the public good to find potential deviators. When \( l \geq 2 \), we design mechanisms in which agents are made to announce an input vector for public goods. Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy. Let \( F^{-1}(x, y) \equiv \{ u' \in U \mid (x, y) \in F(u') \} \) and \( \Lambda^F(x, y) \equiv \cap_{u' \in F^{-1}(x, y)} L((x_i, y), u'_i) \).

**Definition (Condition W*):** For all \( u, u' \in U \) and all \( (x, y) \in F(u) \), if \( \Lambda^F(x, y) \subseteq L((x_i, y), u'_i) \) for all \( i \in I \), then \( (x, y) \in F(u') \).

We define \( I(x, v) \equiv \{ i \in I \mid F^{-1}((\Omega - v - \sum_{j \neq i} x_j, x_{-i}), f(v)) \neq \emptyset \} \).

**Definition (Condition Q):** For every \( (x, v) \in Q_{x_1} \times \cdots \times Q_{x_n} \times Q_v \) such that \( I(x, v) = I \), there exists \( z(x, v) \in A \) such that
(i) \( z_i(x, v) \in \Lambda^F((\Omega - v - \sum_{j \neq i} x_j, x_{-i}), f(v)) \) for all \( i \in I \); and
(ii) if there exists \( u^* \in U \) such that \( \Lambda^F((\Omega - v - \sum_{j \neq i} x_j, x_{-i}), f(v)) \subseteq L(z_i(x, v), u^*_i) \) for all \( i \in I \), then \( z(x, v) \in F(u^*) \).

We obtain the analogy of Proposition 3.1.

**Proposition 4.1:** Let \( n \geq 3 \). An SCC is implementable by a natural quantity mechanism if and only if it satisfies Condition \( W^* \) and Condition \( Q \).

**Proof.** By Proposition 3.1, the proof of necessity is analogous with the case where \( l = 1 \) and \( m = 1 \).
Next, we prove sufficiency by constructing a mechanism.

Rule 1: If for all \( i \in I, s_i = (x_i, v) \), \( I(x, v) = I \), and \((x, f(v)) \in A \), then \( g(s) = (x, f(v)) \)

Rule 2: If for all \( i \in I, s_i = (x_i, v) \), \( I(x, v) = I \), and \((x, f(v)) \notin A \), then \( g(s) = z(x, v) \)

Rule 3: If for all \( i \in I, s_i = (x_i, v) \), and \( 1 \leq \sharp I(x, v) \leq n - 1 \), then

\[
g_i^0(s) = \begin{cases} \Omega/(n - \sharp I(x, v)) & \text{for } i \notin I(x, v) \\ 0 & \text{for } i \in I(x, v) \end{cases} \quad g_i^p(s) = 0
\]

Rule 4: If for some \( i \in I, s_j = (x_j, v^j) \) where \( v^j = v \) for all \( j \neq i, s_i = (x_i, v^i) \) where \( v^i \neq v \), and \( i \in I(x, v) \), then

\[
g(s) = \begin{cases} (x_i, ((\Omega - v^i - x_i)/((n - 1))^j \neq i, f(v^j)) & \text{if } (x_i, f(v^j)) \in A^p \left( ((\Omega - v - \sum_{j \neq i} x_j), x_i), f(v) \right) \\ (\sum_{j \neq i} x_j, f(v)) & \text{otherwise} \end{cases}
\]

Rule 5: For any other case, \( g_i(s) = (\Omega - v^i, f(v^i)) \) and \( g_j(s) = (0, f(v^j)) \) for all \( j \neq i^* \) where \( s_i^* = (x_i^*, v_i^*) \) and \( i^* \) is defined as follows. Let \( \sum_{i \in I} x_{i1} = \alpha \). Since \( x_i \in Q_{x_i}, x_{i1} \in [0, \Omega] \) for all \( i \). Let \( \beta + \gamma = \alpha \), where \( \beta \) is the largest integer less than or equal to \( \alpha \). Then for \( \gamma \in [0, 1) \), there is a unique \( i^* \in I \) such that \( \gamma \in [(i^* - 1)/n, i^*/n) \).

The remain of the proof is analogous with the case where \( l = 1 \) and \( m = 1 \).

To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \). Similarly, we can define natural price\(^a\)-quantity, price\(^b\)-quantity and individual price-quantity mechanisms in \( l \) private goods economies. We show that Q mechanisms implement the Lindahl correspondence in the following subsection.

### 4.2 Implementability of the Lindahl Correspondence

**Lemma 4.1:** Let the Lindahl equilibrium allocation be \((x^L, y^L)\) with the equilibrium price system \((p^L, q^L)\) for some \(u \in U\). If Assumption (C) Assumption (I) are satisfied, then \( \infty > q_i^L > 0 \) for all \( i \in I \), \( \infty > p_j^L > 0 \) and \( \infty > -\partial T/\partial y \bigg|_{y = y^L, v_j = v^L} > 0 \) for all \( j = 1, \ldots, l \), where \( f(v^L) = g^L \).

**Proof.** Let

\[
-\frac{\partial T}{\partial y} \bigg|_{y = y^L, v_j = v^L} = \alpha_1, \ldots, -\frac{\partial T}{\partial y} \bigg|_{y = y^L, v_j = v^L} = \alpha_l.
\]

Since \((x^L, y^L)\) is the Lindahl equilibrium allocation with the equilibrium price system \((p^L, q^L)\) for some \(u \in U\), for all \( i \in I \), the following conditions are satisfied by Assumption (C).

\[
u_i(x^L, y^L) \geq u_i(x_i, y) \quad \text{for all } (x_i, y) \text{ such that } p^L \cdot x_i + q^L y = p^L \cdot \omega_i
\]

Suppose that there exists \( p_i^L \) such that \( p_i^L \leq 0 \). Since Assumption (I) and \( p_i^L \leq 0 \), agent \( i \) can consume \( x_{ij} \) such that \( x_{ij} > x_{ij}^L \). This contradicts (1) by the strict monotonicity of \( u_i \). Next, suppose that \( q_i^L \leq 0 \). Since Assumption (I) and \( q_i^L \leq 0 \), agent \( i \) can consume \( y \) such that \( y > y^L \). This contradicts (1) by the strict monotonicity of \( u_i \). So, \( q_i^L > 0 \) for all \( i \in I \) and \( p_i^L > 0 \) for all \( j = 1, \ldots, l \). Moreover, \( \infty > q_i^L \) for all \( i \in I \) and \( \infty > p_j^L \) for all \( j = 1, \ldots, l \) by Assumption (I). By these facts, we obtain the following conditions.

\[
\frac{q_i^L}{p_i^L} = \frac{\partial u_i}{\partial y} \bigg|_{y = y^L, x_i = x_i^L}, \ldots, \frac{q_i^L}{p_i^L} = \frac{\partial u_i}{\partial x_i} \bigg|_{y = y^L, x_i = x_i^L} = \alpha_1, \ldots, \frac{q_i^L}{p_i^L} = \alpha_l
\]

\[
\sum_{i \in I} q_i^L = \alpha_1, \ldots, \sum_{i \in I} q_i^L = \alpha_l
\]
By these equations, it is clear that \( \infty > \alpha_j = -\frac{\partial T/\partial y}{\partial T/\partial x_{ii}} \bigg|_{v_i = v^L} > 0 \) for all \( j = 1, \ldots, l \). The proof is established.

By this lemma, we can set \( p^L_1 = 1 \) without loss of generality because \( p^L_1 > 0 \). We show the following proposition, using the Lemma 4.1.

**Proposition 4.2:** Let \((x^L, y^L) \in A \) and Assumption (C) and Assumption (I) be satisfied. If there exist some \( u \in U \) such that \((x^L, y^L) \in L(u) \), then the Lindahl equilibrium price system \((p^L, q^L)\) for \((x^L, y^L)\) is unique, where \( p^L_1 = 1 \).

**Proof.** Let

\[
-\frac{\partial T/\partial y}{\partial T/\partial x_{ii}} \bigg|_{v_i = v^L} = \alpha_1, \ldots, -\frac{\partial T/\partial y}{\partial T/\partial x_{ii}} \bigg|_{v_i = v^L} = \alpha_l,
\]

where \( f(v^L) = y^L \). Since \((x^L, y^L)\) is the Lindahl equilibrium allocation with the equilibrium price system \((p^L, q^L)\) for some \( u \in U \), where \( p^L_1 = 1 \), for all \( i \in I \), the following conditions are satisfied by Assumption (C).

\[
\begin{align*}
x^L_{i1} + \sum_{j=2}^{l} p^L_j x^L_{ij} + q^L_{ij} y^L & = \omega_{i1} + \sum_{j=2}^{l} p^L_j \omega_{ij} \quad \text{(4)} \\
q^L_i & = \frac{\partial u_i/\partial y}{\partial u_i/\partial x_{i1}} \bigg|_{x_{i1} = x^L_{i1}} \cdot \ldots \cdot \frac{\partial u_i/\partial y}{\partial u_i/\partial x_{il}} \bigg|_{x_{il} = x^L_{il}} = \frac{\partial u_i/\partial y}{\partial u_i/\partial x_{i1}} \bigg|_{x_{i1} = x^L_{i1}}, \quad \text{(5)} \\
\sum_{i \in I} q^L_i & = \alpha_1, \ldots, \sum_{i \in I} \frac{q^L_i}{p^L_i} = \alpha_l.
\end{align*}
\]

By the Lemma 4.1, \( p^L_j > 0 \) for all \( j = 1, \ldots, l \). Suppose that under \((x^L, y^L)\), there exists the Lindahl equilibrium price system \((p', q')\) for some \( u' \in U \). This price system satisfies the following conditions. For all \( i \in I \),

\[
\begin{align*}
x^L_{i1} + \sum_{j=2}^{l} p^L_j x^L_{ij} + q'_i y^L & = \omega_{i1} + \sum_{j=2}^{l} p^L_j \omega_{ij}, \quad \text{(6)} \\
q'_i & = \frac{\partial u'_i/\partial y}{\partial u'_i/\partial x_{i1}} \bigg|_{x_{i1} = x^L_{i1}} \cdot \ldots \cdot \frac{\partial u'_i/\partial y}{\partial u'_i/\partial x_{il}} \bigg|_{x_{il} = x^L_{il}}, \quad \text{(7)} \\
\sum_{i \in I} q'_i & = \alpha_1, \ldots, \sum_{i \in I} \frac{q'_i}{p'_i} = \alpha_l.
\end{align*}
\]

By (5),

\[
\alpha_1 = \sum_{i \in I} q^L_i = \alpha_2 p^L_2 = \cdots = \alpha_l p^L_l. \quad \text{(8)}
\]

By (7),

\[
\alpha_1 = \sum_{i \in I} q'_i = \alpha_2 p'_2 = \cdots = \alpha_l p'_l. \quad \text{(9)}
\]

Since, by the Lemma 4.1, \( \alpha_1, \ldots, \alpha_l > 0 \), by (8) and (9), \( p^L_1 = p'_2, \ldots, p^L_l = p'_l \). Substituting these for (6), we can obtain the following equation for all \( i \in I \).

\[
x^L_{i1} + \sum_{j=2}^{l} p^L_j x^L_{ij} + q'_i y^L = \omega_{i1} + \sum_{j=2}^{l} p^L_j \omega_{ij} \quad \text{(10)}
\]

By (4) and (10),

\[
q^L_i y^L = q'_i y^L. \quad \text{(11)}
\]
Therefore, since \( q_i^L, q_i^L \in \mathbb{R}_{++} \) by \( m = 1 \) and and \( y^L > 0 \) by assumption (I), we must have \( q_i^L = q_i^L \) for all \( i \in I \). The proof is established. ■

By Proposition 4.2, a social planner can uniquely calculate the Lindahl equilibrium price system for a feasible allocation if he knows each agent’s initial endowment and the marginal rate of transformation at this allocation.

When \( m \geq 2 \), Proposition 4.2 is not true. Let \( m = 2 \), \( y^L = (3, 3) \), \( q_i^L = (\frac{1}{3}, \frac{1}{3}) \) and \( q_i^L = (\frac{1}{6}, \frac{3}{6}) \). Then, equation (11) is satisfied since \( q_i^L \cdot (y^L - \omega_i) = (\frac{1}{3}, \frac{1}{3}) \cdot (3, 3) = 2 \) and \( q_i^L \cdot (y^L - \omega_i) = (\frac{1}{6}, \frac{3}{6}) \cdot (3, 3) = 2 \). But it is clear that \( q_i^L \neq q_i^L \). We will show this counter-example in Example 5.2 (in section 5), when \( l = 1 \).

By Proposition 4.2, \( \Lambda^L_i(x, y) = \Lambda^L_i((x, y), u) \) for all \( i \in I \) and all \( u \in L^{-1}(x, y) \). We obtain the analogy of Proposition 3.6.

**Proposition 4.3:** If Assumption (C) and Assumption (I) are satisfied, then the Lindahl correspondence \( L \) defined on a domain \( U' \equiv \{ u \in U| \omega = (\omega_1, \ldots, \omega_n) \notin L(u) \} \) satisfies Condition \( W^* \) and Condition \( Q \).

**Proof.** The proof is analogous with Proposition 3.6.

Take any \( u \in U \) and \( (x, y) \in L(u) \), where \( y = f(v) \). Then, by Proposition 4.2, the Lindahl equilibrium price \((p, (q_i)_{i \in I}) \in \mathbb{R}_{++}^{I+n}\) for \((x, y) \in L(u)\) is unique since \( m = 1 \) and \( \omega \notin L(u) \) for all \( u \in U' \). So, \((p, q) \in \Psi(y, v, T)\) is satisfied. It is clear that \( H((p, q), (\omega, 0)) = (\omega_i, y) \in \mathbb{R}_{++}^{I+n} \mid p \cdot x_i + q_i y \leq \omega \) and \( x_i + v \leq \Omega \} \subseteq \Lambda^L_i(x, y) \) for all \( i \in I \). If \( \Lambda^L_i(x, y) \subseteq L((x_i, y), u^*_i) \) for all \( i \in I \), then \((x, y) \in L(u^*) \). Therefore \( L \) satisfies Condition \( W^* \).

Let \( z(x, v) = \omega \) for every \((x, v) \in Q_{x_i} \times \cdots \times Q_{x_n} \times Q_{v} \) such that \( I(x, v) = I \). Let \((p^L, q^L) \in \Psi(f(v), v, T)\) be the Lindahl equilibrium price system for an allocation. Since \((p^L, q^L) \in \Psi(f(v), v, T)\) and it is always satisfied that \( \omega_i \in \Lambda^L_i((\Omega - v - \sum_{j \neq i} x_j, x_{-i}), f(v)) \) for all \( i \in I \) by the definition of the SCC \( L \), it is clear that \( L \) satisfies Condition \( Q \). ■

This proposition implies all the four previous mechanisms, i.e., \( Q \), \( IPQ \), \( P^2Q \) and \( P^nQ \) mechanisms implement the Lindahl correspondence when \( m = 1 \).

**Proposition 4.4:** Natural quantity mechanisms implement the Lindahl correspondence.

In one public good economies, a social planner can find each agent’s unique marginal rate of substitution at a feasible allocation by Proposition 4.2 when only each agent’s consumption bundle is announced. Since the marginal rate of substitution conveys local information about preference relations, we can construct mechanisms in which free-riding is undesirable for all agents even if price vectors are not announced.

When \( m \geq 2 \), it is also considered that the public goods are produced one by one rather than at once. But, because a social planner does not know how each agent assigns his own endowment to the cost of each public good, ultimately, that procedure is the same as the one that the plural public goods are produced at once. Also, to produce the public goods one by one, the production function and each agent’s utility function must be separable per public good.

## 5 Economies with \( l \) Private Goods and \( m \) Public Goods

We rewrite the previous model for the economic environments with \( l \) private goods and \( m \) divisible pure public goods. Unlike private goods economies, since there exist personalized prices, the model is somewhat complicated. However, most of the results are analogous with the case where \( l = 1 \) and \( m = 1 \).
Let \(MRT(v, y) \in \mathbb{R}^{nl}_+\) be the matrix of the marginal rate of transformation of \(v\) for \(y\) and \(MRT_{v_j y_k} = -\frac{\partial T/\partial y_k}{\partial T/\partial v_j}\) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\).

\[
\begin{pmatrix}
MRT_{v_1 y_1} & \cdots & MRT_{v_1 y_l} \\
\vdots & \ddots & \vdots \\
MRT_{v_l y_1} & \cdots & MRT_{v_l y_l}
\end{pmatrix}
\]

Then, let a price vector \((p, q) \in \mathbb{R}^{n+m}\) where \(p \in \mathbb{R}_+^l\), \(p_1 = 1\), \(q = (q_1, \ldots, q_n) \in \mathbb{R}^m_+\). If agent \(i\) maximizes his utility in his budget constraint, \(MRS_{x_i y_k} = \frac{\partial u_i/\partial y_k}{\partial u_i/\partial x_i} = \frac{q_k}{p_k}\) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\). We assume that there is no transfer of tax, i.e., \(q_i \in \mathbb{R}^m_+\) for all \(i \in I\). Agents’ shares of costs for a public good \(k\) are \(q_{ik} : \cdots : q_{nk}\) for each \(k = 1, \ldots, m\).

Given \(u_i \in U_i\), let \(D_{ui}(x_i, y)\) be the gradient vector at \((x_i, y)\) which is normalized to belong to the unit simplex \(\Delta \subseteq \mathbb{R}^{n+m}_+.\) We abuse \(\Delta\) and let \(\Delta : \mathbb{R}^l_+ \times \mathbb{R}^m_+ \to \Delta\) and \(\Delta(p, q) \in \Delta\).

**Example 5.1:** Let \(n = 3, l = 2\) and \(m = 1\).

\[
(p, q) = (p_1, p_2, q_1, q_2, q_3) = (1, 2, 1, 1, 1, 1, 6)
\]

\[
\Delta(p, q_1) = \left(\frac{1}{1 + 2 + \frac{1}{2}}, \frac{2}{1 + 2 + \frac{1}{2}}, \frac{1}{1 + 2 + \frac{1}{2}}\right) = \left(\frac{2}{7}, \frac{4}{7}, \frac{1}{7}\right)
\]

\[
\Delta(p, q_2) = \left(\frac{1}{1 + 2 + \frac{1}{2}}, \frac{2}{1 + 2 + \frac{1}{2}}, \frac{1}{1 + 2 + \frac{1}{2}}\right) = \left(\frac{3}{10}, \frac{6}{10}, \frac{1}{10}\right)
\]

\[
\Delta(p, q_3) = \left(\frac{1}{1 + 2 + \frac{1}{6}}, \frac{2}{1 + 2 + \frac{1}{6}}, \frac{1}{1 + 2 + \frac{1}{6}}\right) = \left(\frac{6}{19}, \frac{12}{19}, \frac{19}{19}\right)
\]

We define \(\Psi(y, v, T) = \{(p, q) \in \mathbb{R}^l_+ \times \mathbb{R}^m_+ | \sum_{i \in I} q_{ik} = -\frac{\partial T/\partial y_k}{\partial T/\partial v_j} p_j\} \) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\) and \(\Pi(x, y, v, T, u) = \{(p, q) \in \mathbb{R}^l_+ \times \mathbb{R}^m_+ \mid \Delta(p, q_i) = D_{ui}(x_i, y)\} \) for all \(i \in I\) and \(\sum_{i \in I} q_{ik} = -\frac{\partial T/\partial y_k}{\partial T/\partial v_j} p_j\) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\). Notice that if \((x, y, \epsilon, v, T, u) \subseteq P(u)\) with \((p, q)\), then \((p, q) \in \Pi(x, y, v, T, u)\). Let \(H((p, q_i), (x_i, y))\) = \{\((x'_i, y') \in \mathbb{R}^{l+m}_+ | p \cdot x'_i + q_i \cdot y' \leq p \cdot x_i + q_i \cdot y\) and \((x_i, y) \in Q_i\}\}.

If \((x, y)\) is the Lindahl equilibrium allocation with the price vector \((p, q)\), then \(\sum_{i \in I} MRS_{x_i y_k} = \sum_{i \in I} \frac{\partial u_i/\partial y_k}{\partial u_i/\partial x_i} = \sum_{i \in I} q_{ik} / p_j\) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\). We obtain \(\sum_{i \in I} q_{ik} = -\frac{\partial T/\partial y_k}{\partial T/\partial v_j} p_1 = \cdots = -\frac{\partial T/\partial y_k}{\partial T/\partial v_j} p_l\) for all \(k = 1, \ldots, m\).

### 5.1 Natural Individual Price-Quantity Mechanisms

We begin with natural individual price-quantity mechanisms since IPQ mechanisms can not implement the Lindahl correspondence and the Pareto correspondence in these plural public goods economies. This implies that Q mechanisms do not implement these correspondence. So, it is sufficient to begin with IPQ mechanisms in these economies.

We define terms, before we define the mechanisms. The mechanism \(\Gamma = (S, g)\) is **individually feasible** if \(g(s) \in \mathbb{R}^{l+m}_+\) for all \(s \in S\). The mechanism \(\Gamma = (S, g)\) is **balanced** if \(\sum_{i \in I} g_i'(s) + \sum_{r = 1}^\mu v^k = \Omega\) and \(f(v) = g^\nu(s)\) for all \(s \in S\). Let \(\Psi_i(y, v, T) = \{(p, q_i) \in \mathbb{R}^l_+ \times \mathbb{R}^m_+ | q_{ik} \leq -\frac{\partial T/\partial y_k}{\partial T/\partial v_j} p_j\} \) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\). 

**Definition (Natural Individual Price-Quantity (IPQ) Mechanisms):** The SCC \(F\) is implementable by a **natural individual price-quantity mechanism** if there exists a mechanism \(\Gamma = (S, g)\) such that
(i) \( \Gamma \) implements \( F \);
(ii) for all \( i \in I, S_i = \Psi_i(y, v^i, T) \times Q \);
(iii) for all \( u \in U \) and all \( (x, y) \in F(u) \), if \( \Delta(p, q_i) = Du_i(x_i, y) \) for all \( i \in I, y = f(v) \) and \( s_i = ((p, q_i), (x_i, v)) \) for all \( i \in I \), then \( s \in N_g(u) \) and \( q(s) = (x, y) \);
(iv) \( \Gamma \) is individually feasible and balanced; and
(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces a price vector of private goods, his own personalized price vector of public goods, his own consumption bundle of private goods and an input vector for public goods. Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy. Let \( F^{-1}((x, y), u) = \{ u' \in U | (x, y) \in F(u') \} \) and \( Du_i(x_i, y) = Du_i(x_i, y) \) for all \( i \in I \) and \( \Lambda_i^F((x, y), u) = \cap_{u' \in F^{-1}((x,y),u)} L((x, y), u'_i) \).

**Definition (Condition M):** For all \( u, u' \in U \) and all \( (x, y) \in F(u) \), if \( \Lambda_i^F((x, y), u) \subseteq L((x, y), u'_i) \) for all \( i \in I \), then \( (x, y) \in F(u') \).

Newly, we define \( I^+((p, q), (x, v)) = \{ i \in I | F^{-1}(((\Omega - \sum_{k=1}^m v^k - \sum_{j \neq i} x_j), x_{-i}), f(v)), (\beta_i, 1)MRT(v, f(v)p - \sum_{j \neq i} q_j, q_{-i}) \neq \emptyset \} \).

\[
\frac{1}{I} MRT(v, y)p = \frac{1}{I} \begin{pmatrix}
-\frac{\partial T}{\partial y_i} \,
... \\
\vdots \\
-\frac{\partial T}{\partial y_m} 
\end{pmatrix}
\begin{pmatrix}
p_1 \\
... \\
p_m 
\end{pmatrix}
= \frac{1}{I} \begin{pmatrix}
-\frac{\partial T}{\partial y_i} p_1 + \cdots + (-\frac{\partial T}{\partial y_m} p_1) \\
\vdots \\
-\frac{\partial T}{\partial y_m} p_1 + \cdots + (-\frac{\partial T}{\partial y_m} p_1) 
\end{pmatrix}
\]

Notice that Lindahl-Bowen-Samuelson condition is the necessary condition for an allocation to be Pareto efficient. When \((x, y)\) is Pareto efficient, where \( \sum_{k=1}^m v^k = \Omega - \sum_{i \in I} x_i \) and \( y = f(v) \) since \( \sum_{i \in I} q_k = -\frac{\partial T}{\partial y_i} p_1 = \cdots = -\frac{\partial T}{\partial y_m} p_1 \) for all \( k = 1, \ldots, m \), let \( -\frac{\partial T}{\partial y_i} p_1 = \cdots = -\frac{\partial T}{\partial y_m} p_1 = \beta_k \) for all \( k = 1, \ldots, m \). Then,

\[
\frac{1}{I} MRT(v, y)p = \frac{1}{I} \begin{pmatrix}
l \times \beta_1 \\
... \\
l \times \beta_m 
\end{pmatrix} = \begin{pmatrix}
\beta_1 \\
... \\
\beta_m 
\end{pmatrix}
\]

Let \( q_i = \frac{1}{I} MRT(v, y)p - \sum_{j \neq i} q_j = (\beta_1, \ldots, \beta_m) - \sum_{j \neq i} (q_j, \ldots, q_m) \). Since \( \sum_{i \in I} q_i = (\beta_1, \ldots, \beta_m) \), \( \sum_{i \in I} q_k = \beta_k \) for all \( k = 1, \ldots, m \). This implies \( (p, \frac{1}{I} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) \in \Psi(y, v, T) \). If \((p, q)\) is the Lindahl equilibrium price for \((x, y)\), where \( \sum_{k=1}^m v^k = \Omega - \sum_{i \in I} x_i \) and \( y = f(v) \), then \( q_i = \frac{1}{I} MRT(v, y)p - \sum_{j \neq i} q_j \) for all \( i \in I \) by the LBS condition.

Like Section 3, when we construct the mechanism to implement a Pareto efficient SCC \( F \), the Lindahl-Bowen-Samuelson condition must be satisfied at an equilibrium. In this section, the LBS condition is that \( \sum_{i \in I} q_i = (\beta_1, \ldots, \beta_m) \). Suppose that each agent \( i \) announces \((p, q_i, (x_i, v))\) such that \((x, f(v)) \in A \) and \( \sum_{i \in I} q_i \neq (\beta_1, \ldots, \beta_m) \). If the social planner can find a preference profile \( u \in U \) such that \((x, f(v)) \in F(u) \) and \( \Delta(p, q_j) = Du_j(x_j, f(v)) \) for all \( j \neq i \), he concludes that agent \( i \) should be punished. We call such an agent a potential deviator, since agent \( i \) should have announced \( \frac{1}{I} MRT(v, f(v)p - \sum_{j \neq i} q_j) \) rather than \( q_i \). If a social planner can not find such a preference profile, agents other than agent \( i \) might have announced wrong prices. Therefore, agent \( i \) is not counted as a potential deviator. But, when all agents are potential deviators, we can not assign them zero allocation as punishment since we demand feasibility. Then, we define the following condition.
Definition (Condition IPQ): For every \((p, q), (x, v)\) \(\in \mathbb{R}_+^l \times \mathbb{R}_+^{m+n} \times Q_{x_1} \times \cdots \times Q_{x_n} \times Q_v\) such that \(I^*(p, q), (x, v) = I\), there exists \(z((p, q), (x, v)) \in A\) such that

(i) \(z_i((p, q), (x, v)) \in \Lambda^F_i\left(\left(\left(\Omega - \sum_{k=1}^m v^k - \sum_{j \neq i} x_j\right) x_{i-1}\right), f(v)\right), (p, \frac{1}{\partial T/\partial v_j} MRT(v, f(v))p - \sum_{j \neq i} q_j, q_{i-1})\) for all \(i \in I\); and

(ii) if there exists \(u^* \in U\) such that \(\Lambda^F_i\left(\left(\left(\Omega - \sum_{k=1}^m v^k - \sum_{j \neq i} x_j\right) x_{i-1}\right), f(v)\right), (p, \frac{1}{\partial T/\partial v_j} MRT(v, f(v))p - \sum_{j \neq i} q_j, q_{i-1}) \subseteq L(z_i((p, q), (x, v)), u^*_i)\) for all \(i \in I\), then \(z((p, q), (x, v)) \in F(u^*)\).

We obtain the analogy of Proposition 3.2.

**Proposition 5.1:** Let \(n \geq 3\). Suppose that for all \(i \in I\) and all \(u_i \in U_i\), \(u_i\) is differentiable. Then a Pareto efficient SCC is implementable by a natural individual price-quantity mechanism if and only if it satisfies Condition \(M\) and Condition \(IPQ\).

The proof of Proposition 5.1 is in the appendix. To construct Rule 4 in this mechanism, we need the assumption that \(n \geq 3\).

We can consider a natural quantity mechanism when \(l \geq 2\) and \(m \geq 2\), but even an IPQ mechanism do not implement the Lindahl correspondence. We see this the following subsection.

### 5.2 Natural Price²-Quantity Mechanisms

Let \(\Psi_{\tau}(y, v^i, T) = \{(p, q_i, q_{i+1}) \in \mathbb{R}_+^l \times \mathbb{R}_+^{m+n} | q_{ik} + q_{i+1,k} \leq -\frac{\partial^2 T/\partial u_k \partial v_j}{\partial T/\partial v_j} p_j\) for all \(k = 1, \ldots, m\) and all \(j = 1, \ldots, l\),\}

where we denote \(v_j^i \in \mathbb{R}_+^l\) by input of the private good \(j\) for the public good \(k\) announced by agent \(i\).

Definition (Natural Price²-Quantity (P²Q) Mechanisms): The SCC \(F\) is implementable by a natural price²-quantity mechanism if there exists a mechanism \(\Gamma = (S, g)\) such that

(i) \(\Gamma\) implements \(F\);

(ii) for all \(i \in I\), \(S_i = \Psi_{\tau}(y, v^i, T) \times Q\);

(iii) for all \(u \in U\) and all \((x, y) \in F(u)\), if \(\Delta(p, q_i) = D_{u_i}(x_i, y)\) for all \(i \in I\), \(y = f(v)\) and \(s_i = (p, q_i, q_{i+1}, (x_i, y))\) for all \(i \in I\), then \(s \in N_g(u)\) and \(g(s) = (x, y)\) where \(n + 1 = 1\);

(iv) \(\Gamma\) is individually feasible and balanced; and

(v) \(\Gamma\) satisfies the best response property.

In this mechanism, each agent announces a price vector of private goods, his own personalized price vector of public goods, his neighbor’s personalized price vector of public goods, his own consumption bundle of private goods and an input vector for public goods. Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy.

Newly, we define \(F^{-1}((x, y), (p, q)) \equiv \{u' \in U | (x, y) \in F(u')\}, \Delta(p, q_i) = D_{u_i}(x_i, y)\) for all \(i \in I\) and \((p, q) \in \Psi(y, v, T)\), \(\Lambda^F_i((x, y), (p, q)) \equiv \cap u' \in F^{-1}((x, y), (p, q))L((x_i, y), u'_i), (p, q) \equiv (\{i \in I | F^{-1}(((\left(\left(\Omega - \sum_{k=1}^m v^k - \sum_{j \neq i} x_j\right) x_{i-1}\right), f(v))\right), (p, q) \neq 0\}

**Definition (Condition P²nQ):** For every \((p, q), (x, v)\) \(\in \Psi(f(v), v, T) \times Q_{x_1} \times \cdots \times Q_{x_n} \times Q_v\) such that \(I((p, q), (x, v)) = I\), there exists \(z((p, q), (x, v)) \in A\) such that

(i) \(z_i((p, q), (x, v)) \in \Lambda^F_i\left(\left(\left(\Omega - \sum_{k=1}^m v^k - \sum_{j \neq i} x_j\right) x_{i-1}\right), f(v)\right), (p, q)\) for all \(i \in I\); and

(ii) if there exists \(u^* \in U\) such that \(\Lambda^F_i\left(\left(\left(\Omega - \sum_{k=1}^m v^k - \sum_{j \neq i} x_j\right) x_{i-1}\right), f(v)\right), (p, q) \subseteq L(z_i((p, q), (x, v)), u^*_i)\) for
all \( i \in I \), then \( z((p, q), (x, v)) \in F(u^*) \).

**Definition (P^2Q-Consistency):** Let \( s_i = (p_i', q_i', q_{i+1}', (x_i', v')) \in S_i = \Psi_{i\alpha}(y, v', T) \times Q \), and \( s \in S \), \((p, q) \in \Psi(f(v), v, T)\), and \((x, f(v)) \in A\) be given. We say that \( s \) is P^2Q-consistent with \((p, q)\) and \( v \) if for all \( j \in I \), \( q_j'^{-1} = q_j, \; p_j' = p \) and \( v' = v \). Given \( i \in I \), the strategy profile \( s_{-i} \in S_{-i} \) is P^2Q-consistent with \((p, q)\) and \( v \) if for all \( j \neq i, \; i + 1, \; q_j'^{-1} = q_j, \; q_i'^{-1} = q_i \) and \( q_{i+1}' = q_{i+1} \) and for all \( j \neq i \), \( p_i' = p \) and \( v' = v \).

Like Section 3, when \( n \geq 3 \), in P^2Q mechanisms, a social planner can identify the agent deviating from an equilibrium, using the above notion.

We obtain the analogy of Proposition 3.3.

**Proposition 5.2:** Let \( n \geq 3 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i \), \( u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural price^2-quantity mechanism if and only if it satisfies Condition \( M \) and Condition \( P^nQ \).

The proof of Proposition 5.2 is in the appendix. To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \).

### 5.3 Natural Price^n-Quantity Mechanisms

**Definition (Natural Price^n-Quantity (P^nQ) Mechanisms):** The SCC \( F \) is implementable by a natural price^n-quantity mechanism if there exists a mechanism \( \Gamma = (S, g) \) such that

(i) \( \Gamma \) implements \( F \);

(ii) for all \( i \in I \), \( S_i = \Psi(y, v, T) \times Q \);

(iii) for all \( u \in U \) and all \((x, y) \in F(u)\), if \( \Delta(p, q_i) = Du_i(x, y) \) for all \( i \in I \), \( y = f(v) \) and \( s_i = ((p, q), (x_i, v)) \) for all \( i \in I \), then \( s \in N_{s_i}(u) \) and \( g(s) = (x, y) \);

(iv) \( \Gamma \) is individually feasible and balanced; and

(v) \( \Gamma \) satisfies the best response property.

In this mechanism, each agent announces a price vector of private goods, all agents’ personalized price vectors of public goods, his own consumption bundle of private goods and an input vector for public goods. Condition (iii) is forthrightness. We can say that this mechanism is “natural” in the sense that it satisfies forthrightness, individual feasibility, balancedness, and the best response property which mechanisms should satisfy. We obtain the analogy of Proposition 3.4.

**Proposition 5.3:** Let \( n \geq 3 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i \), \( u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural price^n-quantity mechanism if and only if it satisfies Condition \( M \) and Condition \( P^nQ \).

The proof of Proposition 5.3 is in the appendix. To construct Rule 4 in this mechanism, we need the assumption that \( n \geq 3 \). We rewrite Proposition 3.5 as follows.

**Proposition 5.4:** Let \( n \geq 3 \). Suppose that for all \( i \in I \) and all \( u_i \in U_i \), \( u_i \) is differentiable. Then a Pareto efficient SCC is implementable by a natural price^n-quantity mechanism if and only if it is implementable by a natural price^2-quantity mechanism.
5.4 Implementability of the Lindahl Correspondence

When \( m \geq 2 \), Proposition 4.2 is not true. We show this counter-example in Example 5.2, when \( l = 1 \).

Example 5.2: Let \( n = 3 \), \( l = 1 \) and \( m = 2 \). We assume that \( y_1 = f_1(v^1) = v^1 \), \( y_2 = f_2(v^2) = v^2 \). It is easy to see that \( -\frac{\partial T}{\partial y_1} = -\frac{\partial T}{\partial y_2} = 1 \). Notice that \( v^k = y^k \), where \( f(v^k) = y^L \). We show that there exist two equilibrium price systems such that \( (x^L, y^L) = ((1, 1, 1), (3, 3)) \) \( \in L(u) \) for some \( u \in U \).

First, we consider the following economy \( E_1 = (u^1, \omega^1) = (u^1, (3, 3)) \). For all \( i = 1, 2, 3 \), given the Lindahl equilibrium system \( (p^{L1}, q^{L1}) \), each agent’s problem is

\[
\begin{align*}
\max_{(x, y) \in Q_i} & \quad u^i_1 = \frac{x^1}{y^1} + \frac{1}{2} y^2, \\
\text{subject to} & \quad x_1 + q^{L1}_{11} y_1 + q^{L1}_{12} y_2 = 3, \\
\end{align*}
\]

It is easy to check that in this economy, the Lindahl equilibrium price system for \( (x^L, y^L) = ((1, 1, 1), (3, 3)) \) is \( (p^{L1}, q^{L1}) = (1, (\frac{1}{3}, \frac{1}{3})) \).

Next, we consider the following economy \( E_2 = (u^2, \omega^2) = (u^2, (3, 3, 3)) \). Given the Lindahl equilibrium system \( (p^{L2}, q^{L2}) \), all agents problems are

\[
\begin{align*}
\max_{(x, y) \in Q_2} & \quad u^2_1 = \frac{x^1}{y^1} + \frac{1}{2} y^2, \\
\text{subject to} & \quad x_1 + q^{L2}_{11} y_1 + q^{L2}_{12} y_2 = 3, \\
\max_{(x, y) \in Q_3} & \quad u^2_2 = \frac{x^2}{y^1} + \frac{1}{2} y^2, \\
\text{subject to} & \quad x_2 + q^{L2}_{21} y_1 + q^{L2}_{22} y_2 = 3, \\
\max_{(x, y) \in Q_3} & \quad u^2_3 = \frac{x^3}{y^1} + \frac{1}{2} y^2, \\
\text{subject to} & \quad x_3 + q^{L2}_{31} y_1 + q^{L2}_{32} y_2 = 3.
\end{align*}
\]

It is easy to check that in this economy, the Lindahl equilibrium price system for \( (x^L, y^L) = ((1, 1, 1), (3, 3)) \) is \( (p^{L2}, q^{L2}) = (1, (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})) \).

Therefore, there are two possible economies, \( E_1 \) and \( E_2 \), i.e., two possible price systems \( (p^{L1}, q^{L1}) \) and \( (p^{L2}, q^{L2}) \) for an allocation \( (x^L, y^L) = ((1, 1, 1), (3, 3)) \) \( \in A \) such that \( (x^L, y^L) = ((1, 1, 1), (3, 3)) \) \( \in L(u) \) for some \( u \in U \). In this example, we can say that the planner needs an information of the price system since there are two possible price systems for each agent’s announcement, \( (x^L, y^L) = (x^L, y^L) = ((1, 1, 1), (3, 3)) \) in quantity-mechanisms.

When \( m \geq 2 \), for each agent, there exist plural ways to pay taxes keeping the total of taxes unchanged for each private good because an input vector of each agent is not the component of each agent’s strategy in our mechanisms. So, plural price vectors are possible for an allocation. However, when \( m = 1 \), if for each agent, the total of taxes is kepted unchanged for each private good, there exists a unique way to pay taxes. So, a price vector is uniquely determined. Figure 1 in the appendix represents the case where \( l = 1 \), \( m = 2 \) and \( p = 1 \). Figure 2 in the appendix represents the case where \( l = 2 \), \( m = 1 \) and \( p_1 = 1 \).

Proposition 5.5: Let \( n \geq 3 \). Then, the Lindahl correspondence \( L \) satisfies Condition \( M \) and Condition \( P^nQ \).

Proof. Take any \( u \in U \) and \( (x, y) \in L(u) \), where \( y = f(v) \). Then there exists the Lindahl equilibrium price \( (p, q) \in \Psi(y, v, T) \) for \( (x, y) \) and \( u \). It is clear that for all \( u' \in L^{-1}((x, y), u) \), \( H((p, q_1), \omega_1) = \{(x, y) \in \mathbb{R}_+^{l+m} | p \cdot \omega_1 \geq p \cdot x_1 + q_1 \cdot y \} \subseteq L((x_1, y_1), u'_1) \) for all \( i \in I \). This implies \( \{(x, y) \in \mathbb{R}_+^{l+m} | p \cdot \omega_1 \geq 21 \)
p \cdot x_1 + q_1 \cdot y \) and 
\( x_i + \sum_{k=1}^{m} v^k \leq \Omega \) for all \( i \in I \). If \( \Lambda^I_t((x, y), u) \subseteq L((x_i, y), u^*) \) for all \( i \in I \), then 
\((x, y) \in L(u^*)\). Therefore \( L \) satisfies Condition \( M \).

Let \( z((-p), (x, v)) = \omega \) for every \( ((p, q), (x, v)) \in \Psi(f(v), v, T) \times Q_x \times \cdots \times Q_{x_n} \times Q_y \) such that \( I((p, q), (x, v)) = I \). Since \( (p, q) \in \Psi(f(v), v, T) \) and it is always satisfied that \( \omega_i \in \Lambda^I_t(((\Omega - \sum_{k=1}^{m} v^k - \sum_{j \neq i} x_j), x_{-i}), f(v)), (p, q)) \) for all \( i \in I \) by the definition of SCC \( L \), it is clear that \( L \) satisfies Condition \( P^u Q \).

This proposition implies \( P^Q \) and \( P^u Q \) mechanisms implement the Lindahl correspondence when there exist more than one private goods and more than one public goods.

Unlike the case where \( l = 1 \) and \( m = 1 \), we obtain negative result in this general case. In the case where there exist plural public good, since the Lindahl equilibrium price system for an allocation is not uniquely determined, announcements about price system is needed.

**Example 5.3:** Let \( n = 3 \), \( l = 1 \), \( m = 2 \) and \( \omega_i = 5 \) for all \( i \in I \). We assume that \( y_1 = f_1(v^1) = v^1 \), \( y_2 = f_2(v^2) = v^2 \), \( p, q \in (1, \left(\frac{1}{5}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{5}\right)) \notin \Psi(y, v, T) \) and \( x, f(v) \in (\left(\frac{19}{5}, \frac{19}{5}, \frac{19}{5}\right), (3, 3)) \notin A \). We consider the rich \( U \) such that \( I^*(((p, q), (x, v)) = I \). Define \( Z_i((p, q), (x, v)) = \{(x', y') \in \Omega | (x', y') \in \Lambda^I_t(((\Omega - \sum_{k=1}^{m} v^k - \sum_{j \neq i} x_j), x_{-i}), f(v)), (p, \frac{1}{2} MRT(v, f(v)p - \sum_{j \neq i} q_j, q_{-i})) \) for all \( i \in I \). Since \( \Omega = \omega_1 + \omega_2 + \omega_3 = 15 \), \( v = (3, 3) \), \( \sum_{j \neq i} x_j = \frac{38}{9} \), \( x_{-i} = (\frac{19}{9}, \frac{19}{9}) \) for all \( i, j \in I \) and \( y = f(v) = (3, 3) \),

\[
(((\Omega - \sum_{k=1}^{m} v^k - \sum_{j \neq i} x_j), x_{-i}), y) = (((15 - \frac{38}{9}, 15 \frac{19}{9}, 15 \frac{19}{9})), (3, 3)) = ((\frac{7}{5}, \frac{19}{5}, \frac{19}{5}), (3, 3)).
\]

Since \( p = 1 \), \( t = 1 \), \( MRT_{y_{i1}} = 1 \), \( MRT_{y_{i2}} = 1 \), \( \sum_{j \neq i} q_j = (\frac{2}{10}, \frac{2}{10}) \), and \( q_{-i} = (\frac{1}{10}, \frac{1}{10}) \) for all \( i \in I \),

\[
(p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) = (1, \left(\frac{8}{10}, \frac{8}{10}\right), \left(\frac{1}{10}, \frac{1}{10}\right), \left(\frac{1}{10}, \frac{1}{10}\right)),
\]

\[
(p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) = (1, \left(\frac{1}{10}, \frac{1}{10}\right), \left(\frac{8}{10}, \frac{8}{10}\right), \left(\frac{1}{10}, \frac{1}{10}\right)),
\]

\[
(p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) = (1, \left(\frac{1}{10}, \frac{1}{10}\right), \left(\frac{1}{10}, \frac{1}{10}\right), \left(8 \frac{8}{10}\right)).
\]

In this case, \( Z_i((p, q), (x, v)) = \Psi((-p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}), (\omega_i, 0)) \subseteq \{(x', y') \in \Omega \} \) \( (p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) \leq (x', y') \leq \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) \) \( (\omega_i, 0) \) for all \( i \in I \). Define \( Z((p, q), (x, v)) = \{(x', y') \in A | (x', y') \in Z((p, q), (x, v)) \} \) for all \( i \in I \). Since \( \omega \in Z((p, q), (x, v)) \subseteq A \), \( Z((p, q), (x, v)) = \emptyset \). Pick any \( (x', y') \in Z((p, q), (x, v)) \). Since \( Z_i((p, q), (x, v)) = \Psi((-p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}), (\omega_i, 0)) \) for all \( i \in I \), the normal vector on the hyperplane of agent \( i \)'s budget constraint is \( (1, \left(\frac{8}{10}, \frac{8}{10}\right)) \). We can find some \( u^* \in U \) such that \( \Lambda^I_t(((\Omega - \sum_{k=1}^{m} v^k - \sum_{j \neq i} x_j), x_{-i}), y), (p, \frac{1}{2} MRT(v, y)p - \sum_{j \neq i} q_j, q_{-i}) \subseteq L((x', y), u^*) \) for all \( i \in I \), but the Lindahl-Bowen Samuelson condition is not satisfied because \( \frac{8}{10} + \frac{8}{10} + \frac{8}{10} \neq 1 \). Since we pick \( (x', y') \) from \( Z((p, q), (x, y)) \) arbitrarily, let \( (x', y') = z((p, q), (x, y)) \), and it is clear that \( z((p, q), (x, y)) \notin L(u^*) \). \( L \) does not satisfy Condition \( IPQ \).

To prevent free-riding, announcements about other agents’ price system is needed. That is, the Lindahl correspondence is not implemented by IPQ mechanisms. This is the significant difference between public goods economies and private goods economies. Implementability of the Lindahl correspondence by IPQ mechanisms in public goods economies is parallel to implementability of the Walrasian correspondence by PQ mechanisms in private goods.
economies in terms of strategy spaces but, unfortunately, the Lindahl correspondence is not implemented by IPQ mechanisms.

At a Nash equilibrium, each agent has no incentive to deviate from this equilibrium. So, when the strategy profile \( s \) and \( s_{-i} \) are desirable for a social planner, this planner considers that agent \( i \)'s strategy should be \( s_i \). Then the planner needs to be able to find the deviator \( i \) and punish \( i \) to make \( s \) Nash equilibrium. When \( m \geq 2 \), a social planner can not find the deviator in \( Q \) and IPQ mechanisms because announcements about price vectors are so poor that a planner can not determine each agent’s marginal rate of substitution.

5.5 Implementability of the Pareto Correspondence

We investigate implementability of the Pareto correspondence. We introduce several notions.

Definition (Local Independence, Condition LI (Nagahisa (1991))): For all \( u, u' \in U \) and all \( (x, y) \in F(u) \), if \( Du_i(x, y) = Du'_i(x, y) \) for all \( i \in I \), then \( (x, y) \in F(u') \).

It is shown later that this condition is stronger than Condition \( M \).

Proposition 5.6: The Pareto correspondence \( P \) satisfies Condition LI.

Proof. Take any \( u, u' \in U \) and \( (x, y) \in P(u) \). Let \( Du_i(x, y) = Du'_i(x, y) \) for all \( i \in I \) and \( (p_i, q_i) \) be agent \( i \)'s gradient vector at \( (x, y) \) under \( u \) and \( u' \), where without loss of generality \( p_{i1} = 1 \) and for each \( i \in I \), \( p_i \in \mathbb{R}_{+}^{l_i} \), \( q_i \in \mathbb{R}_{+}^m \). By the Assumption (C) and the Assumption (I), the Lindahl-Bowen-Samuelson condition is the first-order condition of the following Lagrangian function:

\[
\mathcal{L} = \sum_{i \in I} \alpha_i u_i(x, y) + \lambda(0 - T(v, f(v))) + \sum_{j=1}^{l} \mu_j \left( \sum_{i \in I} (\omega_{ij} - x_{ij}) - v_j \right).
\]

Since for all \( i \in I \), \( u_i \) is quasi-concave and \( f \) is concave, the first-order condition is sufficient for the Pareto optimality. Also, MRT\((v, y)\) is unchanged. Therefore, \((x, y) \in P(u')\). ■

Proposition 5.7: Condition LI implies Condition \( M \).

Proof. We assume that an SCC \( F \) satisfies Condition LI, i.e., for all \( u, u' \in U \) and \( (x, y) \in A \), if \( (x, y) \in F(u) \) and \( Du_i(x, y) = Du'_i(x, y) \) for all \( i \in I \), then \( (x, y) \in F(u') \). Moreover, take any \( u, u' \in U \) and \( (x, y) \in F(u) \) and let \( \Lambda^F_i((x, y), u) \subseteq L((x, y), u') \) for all \( i \in I \). Since \( u'_i \) is quasi-concave because \( u'_i \in U \), the set \( X_i = \{ (x'_i, y'_i) \in Q_i \mid u'_i(x'_i, y'_i) \leq u'_i(x'_i, y'_i) \} \) is convex and \( (x'_i, y'_i) \in X_i \). Since \( (x, y) \in \Lambda^F_i((x, y), u) \) for all \( i \in I \) by the definition of \( \Lambda^F_i((x, y), u) \), \( (x_i, y_i) \in X_i \cap \Lambda^F_i((x, y), u) \) for all \( i \in I \). So, for all \( i \in I \), the supporting hyperplane of \( X_i \) at \( (x, y) \) is on the boundary of \( \Lambda^F_i((x, y), u) \) and it has normal vector \( p = Du_i(x, y) \) by the definitions of \( \Lambda^F_i((x, y), u) \) and \( X_i \). This implies that the gradient vector of \( u'_i \) at \( (x, y) \), \( Du'_i(x, y) \) is equivalent to \( Du_i(x, y) \) for all \( i \in I \). Since \( Du_i(x, y) = Du'_i(x, y) \) for all \( i \in I \), \( (x, y) \in F(u') \) by Condition LI. ■

We rewrite Condition \( B \) introduced by Dutta et al. (1995) to apply to public goods economies.

Definition (Condition \( B \)): Suppose \( ((p, q), (x, v)) \in \mathbb{R}_{+}^l \times \mathbb{R}_{+}^{nm} \times \mathbb{R}_{+}^{nl} \times \mathbb{R}_{+}^{lm} \) and \( I(((p, q), (x, v))) = I \), with \( (x_{-i}, x'_i, y) \in F(u') \) and \( (p, q) \in \Pi((x_{-i}, x'_i), y, v, T, u') \) for all \( i \in I \), where \( y = f(v) \). Then, there exists \( (z, y_z) \in A \) such that \( (z, y_z) \in H(((p, q), (x'_i, y))) \) for all \( i \in I \). Moreover, if there exists \( u'_i \in U \) such that for all \( i \in I \),
\( u_i'(z_i, y_z) \geq u_i'(z_i', y_z') \) for all \((z_i', y_z') \in H((p, q_i), (x_i', y)) \cap Q_i\), then \((z, y_z) \in F(u')\).

The Lindahl Correspondence from Equal Division: \( L^{ed}(u) = \{(x, y) \in A | y \leq f(v) \) and \( \exists (p, q) \in \mathbb{R}^{l+n+m}_+ \) such that for all \( i \in I, u_i(x_i', y') > u_i(x_i, y) \) implies that \( p \cdot x_i' + q_i \cdot y' > p \cdot \frac{\Omega}{n} \) and \( \sum_{i \in I} q_i \cdot y - p \cdot \sum_{k=1}^m v_k \geq \sum_{i \in I} q_i \cdot y' - p \cdot \sum_{k=1}^m v_k \) \( \forall (v', y') \in \mathbb{R}^{l+n+m}_+ \) s.t. \( y' \leq f(v') \} \)

**Proposition 5.8**: Condition B is equivalent to Condition \( P^nQ \) for the Pareto correspondence \( P \).

**Proof.** Since \( U \) is rich and the class of admissible utility profiles consisting of utility functions which are differentiable, quasi-concave and strictly monotonic, \( \Lambda \) is rich, there exists \( * \) and \( ** \) and the assumptions of the production sector imply that \( \Pi(x', y, v, T, u) \) for all \( i \in I \), \( u_i(x_i', y') > u_i(x_i, y) \) implies that \( p \cdot x_i' + q_i \cdot y > p \cdot \frac{\Omega}{n} \) and \( \sum_{i \in I} q_i \cdot y - p \cdot \sum_{k=1}^m v_k \geq \sum_{i \in I} q_i \cdot y' - p \cdot \sum_{k=1}^m v_k \) \( \forall (v', y') \in \mathbb{R}^{l+n+m}_+ \) s.t. \( y' \leq f(v') \} \)

**Definition (Anonymity):** A social choice correspondence \( F \) is said to be anonymous if \( (x, y) \in F(u) \) implies that \( \pi(x) \in F(\pi(u)) \) for any permutation \( \pi \) of the agents.

**Proposition 5.9**: If \( F \) is an efficient, anonymous SCC defined over a rich domain that satisfies Condition B, then \( F \subseteq L^{ed} \).

**Proof.** Suppose \( F \) satisfies the hypotheses of the proposition. Since \( F \) is efficient, for every \( u \in U \) and \( (x, y) \in F(u) \), \( \Pi(x, y, v, T, u) \neq \emptyset \). Now, if \( F \) is not contained in \( L^{ed} \), there exists \( u \in U \), \( (x, y) \in F(u) \) and \( j \in I \) such that

\[
 p \cdot x_j + q_j \cdot y < p \cdot \frac{\Omega}{n} \tag{*}
\]

where \( p = 1 \) and \( (p, q) \in \Pi(x, y, v, T, u) \), and \( y = f(v) \). Agent \( j \) will remain fixed for the rest of the proof. Let \( \hat{x}_j = \sum_{i=1}^{n} \frac{x_i}{n} \). For every \( i \in I \), we now define a profile of consumption bundles in which \( i \) receives \( x_j \) and all other agents receive \( \hat{x}_j \). Formally, let \( x^i \in \mathbb{R}^l \) be such that \( x^i = x_j \) and \( x^k_j = \hat{x}_j \) for all \( k \neq i \). Since \( (x, y) \in F(u) \) and \( U \) is rich, there exists \( u' \in U \) such that \( (x^i, y^i) \in F(u') \) and \( (1, q) \in \Pi(x^i, y, v, T, u') \). By anonymity, for all \( i \in I \), there exists \( u' \in U \) such that \( (x^i, y) \in F(u') \) and \( (1, q) \in \Pi(x^i, y, v, T, u') \). Let \( \bar{x} \in \mathbb{R}^l \) be such that \( \bar{x}_i = \hat{x}_j \) for all \( i \in I \). Notice that \( (x_j, \bar{x}_{-i}) = x^i \) for all \( i \in I \). Thus \( I((p, q), (\bar{x}, v)) = I \). By Condition B, there exists \( (z, y_z) \in A \) such that

\[
 p \cdot z_i + q_i \cdot y_z \leq p \cdot x_j + q_i \cdot y \quad \text{for all } i \in I \tag{**}
\]

Now, \( * \), \( ** \) and the assumptions of the production sector imply that \( \sum_{i \in I} p \cdot z_i + \sum_{i \in I} q_i \cdot y_z = p \cdot \sum_{i \in I} z_i + p \cdot \sum_{k=1}^m v_k < p \cdot \Omega \), where \( f(v) = y_z \). But this contradicts \((z, y_z) \in A \) and completes the proof. ■

**Proposition 5.9** implies that the Pareto correspondence does not satisfy Condition \( P^nQ \) by **Proposition 5.8**. As shown in Figure 3, when all agents are potential deviators, we can not construct an allocation to punish them. So, there does not exist an allocation satisfying (i) of Condition \( P^nQ \) for the Pareto correspondence \( P \).

Then, we obtain the following result.
Proposition 5.10: The Pareto correspondence $P$ is not implementable by a natural price-quantity mechanism.

Proof. It is clear by Proposition 5.1, Proposition 5.8 and Proposition 5.9. ■

We consider other four mechanisms with larger strategy spaces than before to implement the Pareto correspondence.

Definition (Natural Price-Quantity$^2$ (P$^2$Q$^2$) Mechanisms): The SCC $F$ is implementable by a natural price$^2$-quantity$^2$ mechanism if there exists a mechanism $\Gamma = (S, g)$ such that

(i) $\Gamma$ implements $F$;
(ii) for all $i \in I$, $S_i = \Psi_i(z(y, v^i, T) \times Q^2_{x_i} \times Q_v$;
(iii) for all $u \in U$ and all $(x, y) \in F(u)$, if $\Delta(p, q_i) = Du_i(x_i, y)$ for all $i \in I$, $y = f(v)$ and $s_i = (p, q_i, q_{i+1}, (x_i, x_{i+1}, v))$ for all $i \in I$, then $s \in N_g(u)$ and $g(s) = (x, y)$ where $n + 1 = 1$;
(iv) $\Gamma$ is individually feasible and balanced; and
(v) $\Gamma$ satisfies the best response property.

Definition (Natural Price-Allocation (P$^2$A) Mechanisms): The SCC $F$ is implementable by a natural price$^2$-allocation mechanism if there exists a mechanism $\Gamma = (S, g)$ such that

(i) $\Gamma$ implements $F$;
(ii) for all $i \in I$, $S_i = \Psi_i(z(y, v^i, T) \times A$;
(iii) for all $u \in U$ and all $(x, y) \in F(u)$, if $\Delta(p, q_i) = Du_i(x_i, y)$ for all $i \in I$, $y = f(v)$ and $s_i = (p, q_i, q_{i+1}, (x_i, x_{i+1}, v))$ for all $i \in I$, then $s \in N_g(u)$ and $g(s) = (x, y)$ where $n + 1 = 1$;
(iv) $\Gamma$ is individually feasible and balanced; and
(v) $\Gamma$ satisfies the best response property.

Definition (Natural Price$^n$-Quantity$^2$ (P$^n$Q$^2$) Mechanisms): The SCC $F$ is implementable by a natural price$^n$-quantity$^2$ mechanism if there exists a mechanism $\Gamma = (S, g)$ such that

(i) $\Gamma$ implements $F$;
(ii) for all $i \in I$, $S_i = \Psi_i(z(f(v), v, T) \times Q^2_{x_i} \times Q_v$;
(iii) for all $u \in U$ and all $(x, y) \in F(u)$, if $\Delta(p, q_i) = Du_i(x_i, y)$ for all $i \in I$, $y = f(v)$ and $s_i = ((p, q), (x_i, x_{i+1}, v))$ for all $i \in I$, then $s \in N_g(u)$ and $g(s) = (x, y)$ where $n + 1 = 1$;
(iv) $\Gamma$ is individually feasible and balanced; and
(v) $\Gamma$ satisfies the best response property.

Definition (Natural Price$^n$-Allocation (P$^n$A) Mechanisms): The SCC $F$ is implementable by a natural price$^n$-allocation mechanism if there exists a mechanism $\Gamma = (S, g)$ such that

(i) $\Gamma$ implements $F$;
(ii) for all $i \in I$, $S_i = \Psi_i(z(f(v), v, T) \times A$;
(iii) for all $u \in U$ and all $(x, y) \in F(u)$, if $\Delta(p, q_i) = Du_i(x_i, y)$ for all $i \in I$, $y = f(v)$ and $s_i = ((p, q), (x_i, x_{i+1}, v))$ for all $i \in I$, then $s \in N_g(u)$ and $g(s) = (x, y)$;
(iv) $\Gamma$ is individually feasible and balanced; and
(v) $\Gamma$ satisfies the best response property.

We need the following notion in constructing a mechanism to implement the Pareto correspondence.

Definition (P$^2$Q$^2$-Consistency): Let $s_i = (p^i, q^i, q^i_{i+1}, (x^i_i, x^i_{i+1}, v^i)) \in S_i = \mathbb{R}^+_+ \times \mathbb{R}^{2 \times m} \times Q^2_{x_i} \times Q_v$, and $s \in S$. 

25
\begin{proof}
\begin{align*}
(p, q) \in \Psi(f(v), v, T), \text{ and } (x, f(v)) \in A \text{ be given}. \text{ We say that } s \text{ is } P^2Q^2\text{-consistent with } (p, q) \text{ and } (x, v) \text{ if } s \text{ is } P^2Q\text{-consistent with } (p, q) \text{ and } v \text{ and for all } j \in I, x_j = x_j^{-1} = x_j. \text{ Given } i \in I, \text{ the strategy profile } s_{-i} \in S_{-i} \text{ is } P^2Q^2\text{-consistent with } (p, q) \text{ and } (x, v) \text{ if } s_{-i} \text{ is } P^2Q\text{-consistent with } (p, q) \text{ and } v \text{ and for all } j \in I, x_j = x_j^{-1} = x_j,\end{align*}

or \(s\) is \(P^2Q\)-consistent with \((p, q)\) and \(v\) and for all \(j \notin i, i + 1, x_j = x_j^{i-1} = x_j, x_j^{i+1} = x_j^{i+1} = x_{i+1}\) or both.

Proposition 5.11: Let \(n \geq 3\). Suppose that for all \(i \in I\) and all \(u_i \in U_i\), \(u_i\) is differentiable. Let \(F\) be a Pareto efficient SCC. Then the following statements are equivalent:

(a) \(F\) satisfies Condition \(M\);
(b) \(F\) is implementable by a natural price\(^2\)-quantity\(^2\) mechanism;
(c) \(F\) is implementable by a natural price\(^2\)-allocation mechanism;
(d) \(F\) is implementable by a natural price\(^n\)-quantity\(^2\) mechanism; and
(e) \(F\) is implementable by a natural price\(^n\)-allocation mechanism.

The proof of Proposition 5.11 is in the appendix.

Proposition 5.12: Let \(n \geq 3\). The Pareto correspondence \(P\) is implementable by a natural price\(^2\)-quantity\(^2\) mechanism.

Proof. It is clear by Proposition 5.6, Proposition 5.7 and Proposition 5.11. \(\blacksquare\)

6 Concluding Remarks

We applied the mechanisms which Saijo, Tatamitani and Yamato (1996) constructed to public goods economies when the number of agents is larger than or equal to three. We have find that the performance of mechanisms changes as to the number of public goods and that the class of Pareto efficient SCCs implemented by natural price\(^n\)-quantity mechanisms is equivalent to that of Pareto efficient SCCs implemented by natural price\(^2\)-quantity mechanisms. The Lindahl correspondence is implemented by all natural quantity, individual price-quantity, price\(^2\)-quantity and price\(^n\)-quantity mechanisms in the economies with one public good, but in the economies with more than one public goods, it is not implemented by quantity mechanisms and individual price-quantity mechanisms, because the Lindahl equilibrium price system is not uniquely determined at an allocation when there exist more than one public goods.

We show open problems. First, it is not clear what is the difference between natural mechanisms and elementary mechanisms constructed by Dutta, Sen and Vohra (1995) in public goods economies. In an economy with one private good and one public good, Dutta et al. (1995) constructed a elementary mechanism implementing the Lindahl correspondence. This is a similar result to our research in this economy.

Second, we used moduler construction in our mechanisms. We need to consider whether or not moduler construction is reasonable as the construction of a natural mechanism.

Third, we do not consider the way to choose the “best” mechanism for a society. We must consider what the “best” mechanism is and how we choose the best mechanism. Saijo and Yamato (1999) researched the participation in a mechanism.

Finally, we need to consider the validity of our model. In our model, the production sector is not considered explicitly and there exists neither transfer of tax nor indivisible good. We must solve these problems in further research.
References


**Appendix**

**Proof of Proposition 3.1.** We prove necessity. Suppose that $F$ is implementable by a Q mechanism $\Gamma = (S, g)$. First, we show that $F$ satisfies Condition $W^*$. Let $u \in U$ and $(x, y) \in F(u)$ be given. For each $i \in I$, let $s_i = (x_i, y)$. Then for all $u' \in F^{-1}(x, y)$, $s \in N_g(u')$ and $g(s) = (x, y)$ by forthrightness. This implies that
First, we show that $I^i(S_i, s_{-i}) \subseteq \Lambda_i^F((x, y), u_i')$ for all $i$ and all $u' \in F^{-1}(x, y)$. Thus, $g_i(S_i, s_{-i}) \subseteq \Lambda_i^F((x, y), u_i')$ for all $i \in I$. If $\Lambda_i^F((x, y), u_i') \subseteq L((x, y), u_i')$ for all $i \in I$, then $s \in N_g(u)$ and $g(s) = (x, y) \in g \circ N_g(u') = F(u')$.

Second, we show that $F$ satisfies Condition $Q$. Let $(x, y) \in Q_{x_1} \times \cdots \times Q_{x_n} \times Q_y$ such that $I(x, y) = I$ be given. Then for all $i$ and all $u' \in F^{-1}(((\Omega^i - y - \sum_{j \neq i} x_j), x_{-i}), y)$, $s_i' \in N_g(u')$ and $g(s_i') = (((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y)$, where $s_{i1} = (\Omega - y - \sum_{j \neq i} x_j, y)$ and $s_{i2} = (x_{-i}, y)$ for all $j \neq i$ by forthrightness. This implies that $g_i(S_i, s_{-i}') \subseteq L((\Omega - y - \sum_{j \neq i} x_j), u_i')$ for all $i$ and all $u' \in F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y)$. Thus $g_i(S_i, s_{-i}') \subseteq \Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y)$ for all $i$. Let $z(x, y) = g((x, y) \in I)$. Then $z_i(x, y) \in \Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y)$ for all $i \in I$. Further, suppose that there exists $u^* \in U$ such that $\Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) \subseteq L(z_i(x, y), u_i^*)$ for all $i \in I$. Then $g_i(S_i, s_{-i}') \subseteq L(z_i(x, y), u_i^*)$ for all $i \in I$. Hence $z(x, y) \in g \circ N_g(u') = F(u^*)$.

Next, we prove sufficiency by constructing a mechanism.

Rule 1: If for all $i \in I$, $s_i = (x_i, y_i)$, $I(x, y) = I$, and $(x, y) \in A$, then $g(s) = (x, y)$

Rule 2: If for all $i \in I$, $s_i = (x_i, y_i)$, $I(x, y) = I$, and $(x, y) \notin A$, then $g(s) = (z, x, y)$

Rule 3: If for all $i \in I$, $s_i = (x_i, y_i)$, and $1 \leq z \in I(x, y) \leq n - 1$, then

$$g_i^z(s) = \begin{cases} \Omega/(n - z)(I(x, y)) & \text{for } i \notin I(x, y) \\ 0 & \text{for } i \in I(x, y) \end{cases}$$

Rule 4: If for some $i \in I$, $s_j = (x_j, y_j')$ where $y_j' = y$ for all $j \neq i$, $s_i = (x_i, y_i')$ where $y_i' \neq y$, and $i \in (I, x, y)$, then

$$g(s) = \begin{cases} (x_i, ((\Omega - y - x_i)/(n - 1), y_i') & \text{if } (x_i, y_i') \in \Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) \\ ((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) & \text{otherwise} \end{cases}$$

Rule 5: For any other case, $g_i(s) = ((\Omega - y, y')$ and $g_j(s) = (0, y')$ for all $j \neq i$ where $s_i = (x_i, y')$ and $i^*$ is defined as follows. Let $\sum_{j \in I} x_j = \alpha$. Since $x_i \in Q_{x_i}, x_i \in [0, \Omega]$ for all $i$. Let $\beta + \gamma = \alpha$, where $\beta$ is the largest integer less than or equal to $\alpha$. Then for $\gamma \in [0, 1)$, there is a unique $i^* \in I$ such that $\gamma \in [(i^* - 1)/n, i^*/n)$.

First, we prove that for all $u \in U$, $F(u) \subseteq g \circ N_g(u)$. Pick any $(x, y) \in F(u)$. For each $i$, let $s_i = (x_i, y_i)$. By Rule 1, $g(s) = (x, y)$. By Rule 2-4, $g_i(S_i, s_{-i}) = \Lambda_i^F((x, y), u_i) \subseteq L((x, y), u_i)$ for all $i \in I$. Since $s \in N_g(u)$, $g(s) = (x, y) \in g \circ N_g(u)$.

Second, we show that for all $u^* \in U$, $g \circ N_g(u^*) \subseteq F(u^*)$. Let $s \in N_g(u^*)$ be given. It is easy to see that $s$ can not correspond to Rule 3, Rule 4, or Rule 5 by assumption (D). Suppose that $s$ corresponds to Rule 1. Then, by $I(x, y) = I$ and $(x, y) \in A$, $g(s) = (x, y) \in F(u)$ for some $u \in U$. By Rule 4, $\Lambda_i^F((x, y), u_i) \subseteq g_i(S_i, s_{-i}) \subseteq L((x, y), u_i^*)$ for all $i \in I$. By Condition $W^*$, $(x, y) \in F(u^*)$.

Next suppose that $s$ corresponds to Rule 2. Then $g(s) = z(x, y)$. By Rule 4, $\Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y) \subseteq g_i(S_i, s_{-i}) \subseteq L(z_i(x, y), u_i^*)$ for all $i \in I$. By Condition $Q$, $z(x, y) \in F(u^*)$.

Finally, it is clear that the constructed mechanism satisfies the best response property. ■

Proof of Proposition 3.2. We prove necessity. Suppose that $F$ is implementable by an IPQ mechanism $\Gamma = (S, g)$.

First, we show that $F$ satisfies Condition $M$. Let $u \in U$ and $(x, y) \in F(u)$ be given. Since $F(u') \subseteq F(u)$ for all $u' \in F^-(x, y), u)$, there exists $q \in \Delta \subseteq \mathbb{R}^n$ such that $q_i = Du_i(x, y)$ for all $i \in I$ and all $u' \in F^-(x, y, u)$. For each $i \in I$, let $s_i = (q_i, (x_i, y_i))$. Then for all $u' \in F^-(x, y, u)$, $s_i \in N_g(u')$ and $g(s) = (x, y)$ by forthrightness. This implies that $g_i(S_i, s_{-i}) \subseteq L((x, y), u_i')$ for all $i$ and all $u' \in F^-(x, y, u)$. Thus, $g_i(S_i, s_{-i}) \subseteq \Lambda_i^F((x, y), u_i')$ for all $i \in I$. If $\Lambda_i^F((x, y), u_i') \subseteq L((x, y), u_i')$ for all $i$, then $s \in N_g(u^*)$ and $g(s) = (x, y) \in g \circ N_g(u^*) = F(u^*)$.

Second, we show that $F$ satisfies Condition $IPQ$. Let $(q, (x, y)) \in [0, 1] \times Q_{x_1} \times \cdots \times Q_{x_n} \times Q_y$ such that $P^*(q, (x, y)) = I$ be given. Then for all $i$ and all $u' \in F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i}))$, $s_i \in N_g((u'))$ and $g(s_i') = (((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y)$, where $s_{i1} = (1 - \sum_{j \neq i} q_j, (\Omega - y - \sum_{j \neq i} x_j), x_{-i})$ and $s_{i2} = (q_j, (x, y))$ for all $j \neq i$ by forthrightness. This implies that $g_i(S_i, s_{-i}') \subseteq L((\Omega - y - \sum_{j \neq i} x_j), u_i')$ for all $i$ and all $u' \in F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i}))$. Thus $g_i(S_i, s_{-i}') \subseteq \Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i}))$ for all $i \in I$. If $\Lambda_i^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i})) \subseteq L((x, y), u_i')$ for all $i$, then $s \in N_g(u^*)$ and $g(s) = (x, y) \in g \circ N_g(u^*) = F(u^*)$.
all i. Let \( z(q, (x, y)) = g((q, (x_i, y)))_{i \in I} \). Then \( z_i(q, (x, y)) \in \Lambda_F^s((((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i})) \) for all \( i \in I \). Further, suppose that there exists \( u^* \in U \) such that \( \Lambda_F^s((((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i})) \subseteq L(z_i(q, (x, y)), u_i^*) \) for all \( i \in I \). Then \( g_i(S_i, s_{i-1}^i) \subseteq L(z_i(q, (x, y)), u_i^*) \) for all \( i \in I \). Hence \( z(q, (x, y)) \in g \circ N_g(u^*) = F(u^*) \).

Next, we prove sufficiency by constructing a mechanism.

Rule 1: If for all \( i \in I, s_i = (q_i, (x_i, y_i)) \), \( \sum_{i \in I} q_i = 1, I^*(q, (x, y)) = I \), and \( (x, y) \in A \), then \( g(s) = (x, y) \).

Rule 2: If for all \( i \in I, s_i = (q_i, (x_i, y_i)), I^*(q, (x, y)) = I \), and \( \sum_{i \in I} q_i \neq 1 \) or \( (x, y) \notin A \), then \( g(s) = z(q, (x, y)) \).

Rule 3: If for all \( i \in I, s_i = (q_i, (x_i, y_i)) \) and \( 1 \leq \#I^*(q, (x, y)) \leq n - 1 \), then

\[
g_i^s(s) = \begin{cases} \Omega/((n - 1)I^*(q, (x, y))) & \text{for } i \notin I^*(q, (x, y)) \\ 0 & \text{for } i \in I^*(q, (x, y)) \end{cases}
\]

Rule 4: If for some \( i \in I, s_j = (q_j, (x_j, y_j')) \) where \( y_j' = y \) for all \( j \neq i \), \( s_i = (q_i, (x_i, y_i')) \) where \( y_i' \neq y \), and \( i \in I^*(q, (x, y)) \), then

\[ g(s) = \begin{cases} (x_i, ((\Omega - y - x_i)/(n - 1)), y_i') & \text{if } (x_i, y_i') \in \Lambda_F^s((((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i})) \\ ((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y) & \text{otherwise} \end{cases}
\]

Rule 5: For any other case, \( g^s_i(s) = (\Omega - y^*, y^*) \) and \( g_i(s) = (0, y^*) \) for all \( j \neq i \) where \( s_i = (q_i^*, (x_i^*, y_i^*)) \) and \( i^* \) is defined as follows. Let \( \sum_{i \in I} x_i = \alpha \). Since \( x_i \in Q_{x_i}, x_i \in [0, \Omega] \) for all \( i \). Let \( \beta + \gamma = \alpha \), where \( \beta \) is the largest integer less than or equal to \( \alpha \). Then for \( \gamma \in (0, 1) \), there is a unique \( i^* \in I \) such that \( \gamma \in ([i^* - 1]/n, i^*/n) \).

First, we prove that for all \( u \in U, F(u) \subseteq g \circ N_g(u) \). Pick any \((x, y) \in F(u)\). Then \( \sum_{i \in I} du_i(x_i, y_i) = 1 \) because \( F(u) \subseteq P(u) \) for all \( u' \in U \). For each \( i \), let \( s_i = (q_i, (x_i, y_i)) \) where \( q_i = du_i(x_i, y_i) \) for all \( i \in I \). By Rule 1, \( g(s) = (x, y) \). By Rule 2-4, \( g_i(S_i, s_{i-1}) = \Lambda_F^s((x, y), q) \subseteq L((x, y), u_i) \) for all \( i \in I \). Since \( s \in N_g(u) \), \( g(s) = (x, y) \in g \circ N_g(u) \).

Second, we show that for all \( u^* \in U, g \circ N_g(u^*) \subseteq F(u^*) \). Let \( s \in N_g(u^*) \) be given. It is easy to see that \( s \) can not correspond to Rule 3, Rule 4, or Rule 5 by assumption (D). Suppose that \( s \) corresponds to Rule 1. Then, by \( \sum_{i \in I} q_i = 1, I^*(q, (x, y)) = I \) and \((x, y) \in A \), \( g(s) = (x, y) \in F(u) \) for some \( u \in U \). Notice that by \( \sum_{i \in I} q_i = 1 \) and \( I^*(q, (x, y)) = I, q_i = du_i(x_i, y_i) \) for all \( i \in I \). By Rule 4, \( \Lambda_F^s((x, y), q) \subseteq L((x, y), u_i) \) for all \( i \in I \). By Condition \( M, (x, y) \in F(u^*) \).

Next suppose that \( s \) corresponds to Rule 2. Then \( g(s) = z(q, (x, y)) \). By Rule 4, \( \Lambda_F^s((((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), (1 - \sum_{j \neq i} q_j, q_{-i})) \subseteq g_i(S_i, s_{i-1}) \subseteq L(z_i(q, (x, y)), u_i^*) \) for all \( i \in I \). By Condition \( IPQ, z(q, (x, y)) \in F(u^*) \).

Finally, it is clear that the constructed mechanism satisfies the best response property. ■

**Proof of Proposition 3.3.** We prove necessity, but by Proposition 2, this is clear. Suppose that \( F \) is implementable by a P^2Q mechanism \( \Gamma = (S, g) \). First, we show that \( F \) satisfies Condition \( M \). Let \( u \in U \) and \((x, y) \in F(u) \) be given. Since \( F(u') \subseteq P(u') \) for all \( u' \in F^{-1}((x, y), u) \), there exists \( q \in \Delta \subseteq R^n_u \) such that \( q_i = du_i(x_i, y_i) \) for all \( i \in I \) and all \( u' \in F^{-1}((x, y), u) \). For each \( i \in I \), let \( s_i = (q_i, (x_i, y_i)) \). Then for all \( u' \in F^{-1}((x, y), u) \), \( s_i \in N_g(u') \) and \( g(s) = (x, y) \) by forthrightness. This implies that \( g_i(S_i, s_{i-1}) \subseteq L((x, y), u_i^*) \) for all \( i \) and all \( u' \in F^{-1}((x, y), u) \). Thus, \( g_i(S_i, s_{i-1}) \subseteq \Lambda_F^s((x, y), u_i^*) \) for all \( i \in I \). If \( \Lambda_F^s((x, y), u_i^*) \subseteq L((x, y), u_i^*) \) for all \( i \in I \), then \( s \in N_g(u^*) \) and \( g(s) = (x, y) \in g \circ N_g(u^*) = F(u^*) \).

Second, we show that \( F \) satisfies Condition \( P^2Q \). Let \((q, (x, y)) \in \Delta \times Q_{x_1} \times \cdots \times Q_{x_n} \times Q_y \) such that \( I(q, (x, y)) = I \) be given. Then for all \( i \) and all \( u' \in F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), q) \), \( s_i^j \in N_g(u') \) and \( g(s^i) = (((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \), \( s_i^j \in N_g(u') \) and \( g(s^i) = (((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \) for all \( j \neq i \) by forthrightness. This implies that \( g_i(S_i, s_{i-1}^j) \subseteq L(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), u_i^*) \) for all \( i \) and all \( u' \in F^{-1}(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \). Thus \( g_i(S_i, s_{i-1}^j) \subseteq \Lambda_F^s(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \) for all \( i \). Let \( z(q, (x, y)) = \)}
Rule 1: If $s$ is $P^2Q$-consistent with $q$ and $y$, $\sum_{i \in I} q_i = 1$, $I(q, (x, y)) = I$, and $(x, y) \in A$, then $g(s) = (x, y)$.

Rule 2: If $s$ is $P^2Q$-consistent with $q$ and $y$, $\sum_{i \in I} q_i = 1$, $I(q, (x, y)) = I$, and $(x, y) \notin A$, then $g(s) = z(q, (x, y))$.

Rule 3: If $s$ is $P^2Q$-consistent with $q$ and $y$, $\sum_{i \in I} q_i = 1$ and $1 \leq \sharp I(q, (x, y)) \leq n - 1$, then

$$g^R_i(s) = \begin{cases} \Omega/\Phi(q, (x, y)) & \text{for } i \notin I(q, (x, y)) \\ 0 & \text{for } i \in I(q, (x, y)) \end{cases}$$

Rule 4: If for some $i \in I$, $s_{-i}$ is $P^2Q$-consistent with $q$ and $y$ where $\sum_{i \in I} q_i = 1$, $s_i = (q_i, q_{i+1}, (x, y'))$ where $q_i' \neq q_i$, $q_i \neq q_{i+1}$ or $y' \neq y$, and $i \in I(q, (x, y))$, then

$$g(s) = \begin{cases} (x, y') & \text{if } (x, y') \in \Lambda^F\big(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y)\big), q) \\ (((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y)) & \text{otherwise} \end{cases}$$

Rule 5: For any other case, $g_i(s) = (\Omega - y^*, y^*)$ and $g_j(s) = (0, y^*)$ for all $j \neq i$ where $s_j = (q_j^*, q_j^*+1, (x_j, y_j^*))$ and $i^*$ is defined as follows. Let $\sum_{i \in I} x_i = \alpha$. Since $x_i \in Q_{x_i}, x_i \notin [0, \Omega]$ for all $i$. Let $\beta + \gamma = \alpha$, where $\beta$ is the largest integer less than or equal to $\alpha$. Then for $\gamma \in [0, 1)$, there is a unique $i^* \in I$ such that $\gamma \in [(i^* - 1)/n, i^*/n)$.

First, we prove that for all $u \in U$, $F(u) \subseteq g \circ N_g(u)$. Pick any $(x, y) \in F(u)$. For each $i$, let $s_i = (q_i, q_{i+1}, (x_i, y))$ where $q_i = D_u(x_i, y)$ for all $i \in I$. By Rule 1, $g(s) = (x, y)$. By Rule 2-4, $g_i(S_i, s_{-i}) = \Lambda^F((x, y), q) \subseteq L((x, y), u_i)$ for all $i \in I$. Since $s \in N_g(u)$, $g(s) = (x, y) \in g \circ N_g(u)$.

Second, we show that for all $u^* \in U$, $g \circ N_g(u^*) \subseteq F(u^*)$. Let $s \in N_g(u^*)$ be given. It is easy to see that $s$ cannot correspond to Rule 3, Rule 4, or Rule 5 by assumption (D). Suppose that $s$ corresponds to Rule 1. Then, by $I(q, (x, y)) = I$ and $(x, y) \in A$, $g(s) = (x, y) \in F(u)$ for some $u \in U$. Notice that by $I(q, (x, y)) = I$, $q_i = D_u(x_i, y)$ for all $i \in I$. By Rule 4, $\Lambda^F((x, y), q) = \Lambda^F((x, y), u) \subseteq g_i(S_i, s_{-i}) \subseteq L((x, y), u_i^*)$ for all $i \in I$. By Condition $M$, $(x, y) \in F(u^*)$.

Next suppose that $s$ corresponds to Rule 2. Then $g(s) = z(q, (x, y))$. By Rule 4, $\Lambda^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y), q) \subseteq g_i(S_i, s_{-i}) \subseteq L(z_i(q, (x, y)), u_i^*)$ for all $i \in I$. By Condition $P^0Q$, $z(q, (x, y)) \in F(u^*)$.

Finally, it is clear that the constructed mechanism satisfies the best response property.

**Proof of Proposition 3.4.** We prove necessity. Suppose that $F$ is implementable by a $P^0Q$ mechanism $\Gamma = (S, g)$. First, we show that $F$ satisfies Condition $M$. Let $u \in U$ and $(x, y) \in F(u)$ be given. Since $F(u') \subseteq P(u')$ for all $u' \in F^-(x, y, u)$, there exists $q \in \Delta \subseteq \mathbb{R}^n_+$ such that $q_i = D_{u_i}(x_i, y)$ for all $i \in I$ and all $u' \in F^-(x, y, u)$. For each $i \in I$, let $s_i = (q_i, (x_i, y))$. Then for all $u' \in F^-(x, y, u)$, $s \in N_g(u')$ and $g(s) = (x, y)$ by forthrightness. This implies that $g(S_i, s_{-i}) \subseteq L((x, y), u_i')$ for all $i$ and all $u' \in F^-(x, y, u)$. Thus, $g(S_i, s_{-i}) \subseteq \Lambda^F((x, y), u)$ for all $i \in I$. If $\Lambda^F((x, y), u) \subseteq L((x, y), u_i^*)$ for all $i \in I$, then $s \in N_g(u^*)$ and $g(s) = (x, y) \in g \circ N_g(u^*) = F(u^*)$.

Second, we show that $F$ satisfies Condition $P^0Q$. Let $(x, y) \in \Delta \times Q_{x_i} \times \cdots \times Q_{x_k} \times Q_y$ such that $I(q, (x, y)) = I$ be given. Then for all $i$ and all $u' \in F^--(\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y, q)$, $s' \in N_g(u')$ and $g(s') = (((\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y), q), s'_i = (q_i, (\Omega - y - \sum_{j \neq i} x_j, y))$ and $s'_i = (q_i, (x_i, y))$ for all $j \neq i$ by forthrightness. This implies that $g_i(S_i, s_{-i}^i) \subseteq L((\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y), u_i^i)$ for all $i$ and all $u' \in F^--(\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y, q)$. Thus $g_i(S_i, s_{-i}^i) \subseteq \Lambda^F(((\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y), q) \subseteq L(z_i(q, (x, y)), u_i^*)$ for all $i \in I$. Let $z(q, (x, y)) = g(q, (x, y))$. Then $z_i(q, (x, y)) \in \Lambda^F(((\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y), q) \subseteq L(z_i(q, (x, y)), u_i^*)$ for all $i \in I$. Further, suppose that there exists $u^* \in U$ such that $\Lambda^F(((\Omega - y - \sum_{j \neq i} x_j, x_{-i}, y), q) \subseteq L(z_i(q, (x, y)), u_i^*)$ for all $i \in I$. Then $g_i(S_i, s_{-i}^i) \subseteq L(z_i(q, (x, y)), u_i^*)$ for all $i \in I$. Hence $z(q, (x, y)) \in g \circ N_g(u^*) = F(u^*)$. 


Next, we prove sufficiency by constructing a mechanism.

Rule 1: If for all \( i \in I \), \( s_i = (q_i, (x_i, y_i)) \), \( \sum_{i \in I} q_i = 1 \), \( I(q, (x, y)) = I \), and \( (x, y) \in A \), then \( g(s) = (x, y) \).

Rule 2: If for all \( i \in I \), \( s_i = (q_i, (x_i, y_i)) \), \( \sum_{i \in I} q_i = 1 \), \( I(q, (x, y)) = I \), and \( (x, y) \notin A \), then \( g(s) = z(q, (x, y)) \).

Rule 3: If for all \( i \in I \), \( s_i = (q_i, (x_i, y_i)) \), \( \sum_{i \in I} q_i = 1 \) and \( 1 \leq \sharp I(q, (x, y)) \leq n - 1 \), then
\[
g^x_i(s) = \begin{cases} 
\Omega((n - \sharp I(q, (x, y)))) & \text{for } i \notin I(q, (x, y)) \\
0 & \text{for } i \in I(q, (x, y)) 
\end{cases}
\quad g^y_i(s) = 0
\]

Rule 4: If for some \( i \in I \), \( s_j = (q^j, (x_j, y_j)) \) where \( q^j = q, y^j = y \) and \( \sum_{i \in I} q_i = 1 \) for all \( j \neq i \), \( s_i = (q^i, (x_i, y^i)) \) where \( q^i \neq q \) or \( y^i \neq y \), and \( i \notin I(q, (x, y)) \), then
\[
g(s) = \begin{cases} 
(x_i, ((\Omega - y^i - x_i)/(n - 1))_{j \neq i}, y^j) & \text{if } (x_i, y^j) \in \Lambda^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), q) \\
((\Omega - y - \sum_{j \neq i} x_j), x_{-i}, y) & \text{otherwise}
\end{cases}
\]

Rule 5: For any other case, \( g^\ast_i(s) = (\Omega - y^\ast_i, y^\ast_i) \) and \( g_j(s) = (0, y^\ast_j) \) for all \( j \neq i^\ast \) where \( s_i^\ast = (q^i, (x_i, y^i)) \) and \( i^\ast \) is defined as follows. Let \( \sum_{i \in I} x_i = \alpha \). Since \( x_i \in [0, \Omega] \) for all \( i \). Let \( \beta + \gamma = \alpha \), where \( \beta \) is the largest integer less than or equal to \( \alpha \). Then for \( \gamma \in [0, 1) \), there is a unique \( i^\ast \in I \) such that \( \gamma \in ([i^\ast - 1]/n, i^\ast/n) \).

First, we prove that for all \( u \in U \), \( F(u) \subseteq g \circ N_g(u) \). Pick any \( (x, y) \in F(u) \). For each \( i \), let \( s_i = (q_i, (x_i, y_i)) \) where \( q_i = Du_i(x_i, y) \) for all \( i \in I \). By Rule 1, \( g(s) = (x, y) \). By Rule 2-4, \( g_i(S_i, s_{-i}) = \Lambda^F((x_i, y_i), q) \subseteq L ((x_i, y), u_i) \) for all \( i \in I \). Since \( s \in N_g(u) \), \( g(s) = (x, y) \in g \circ N_g(u) \).

Second, we show that for all \( u^\ast \in U \), \( g \circ N_g(u^\ast) \subseteq F(u^\ast) \). Let \( s \in N_g(u^\ast) \) be given. It is easy to see that \( s \) can not correspond to Rule 3, Rule 4, or Rule 5 by assumption (D). Suppose that \( s \) corresponds to Rule 1. Then, by \( I(q, (x, y)) = I \) and \( (x, y) \in A \), \( g(s) = (x, y) \in F(u) \) for some \( u \in U \). Notice that by \( I(q, (x, y)) = I \), \( q_i = Du_i(x_i, y_i) \) for all \( i \in I \). By Rule 4, \( \Lambda^F((x, y), q) = \Lambda^F((x, y), u) \subseteq g_i(S_i, s_{-i}) \subseteq L ((x, y), u_i) \) for all \( i \in I \). By Condition M, \( (x, y) \in F(u^\ast) \).

Next suppose that \( s \) corresponds to Rule 2. Then \( g(s) = z(q, (x, y)) \). By Rule 4, \( \Lambda^F(((\Omega - y - \sum_{j \neq i} x_j), x_{-i}), y), q) \subseteq g_i(S_i, s_{-i}) \subseteq L(z_i(q, (x, y)), u_i^\ast) \) for all \( i \in I \). By Condition P\text{M}Q, \( z(q, (x, y)) \in F(u^\ast) \).

Finally, it is clear that the constructed mechanism satisfies the best response property.

\textbf{Proof of Proposition 5.1.} By Proposition 3.2, the proof of necessity is analogous with the case where \( l = 1 \) and \( m = 1 \).

Next, we prove sufficiency by constructing a mechanism.

Rule 1: If for all \( i \in I \), \( s_i = ((p, q_i), (x_i, v_i)) \), \( (p, q) \in \Psi(f(v), v, T) \), \( I^\ast((p, q), (x, v)) = I \), and \( (x, f(v)) \in A \), then \( g(s) = (x, f(v)) \).

Rule 2: If for all \( i \in I \), \( s_i = ((p, q_i), (x_i, v_i)) \), \( I^\ast((p, q), (x, v)) = I \), and \( (p, q) \notin \Psi(f(v), v, T) \) or \( (x, f(v)) \notin A \), then \( g(s) = z((p, q), (x, v)) \).

Rule 3: If for all \( i \in I \), \( s_i = ((p, q_i), (x_i, v_i)) \) and \( 1 \leq \sharp I^\ast((p, q), (x, v)) \leq n - 1 \), then
\[
g^x_i(s) = \begin{cases} 
\Omega/((n - \sharp I^\ast((p, q), (x, v)))) & \text{for } i \notin I^\ast((p, q), (x, v)) \\
0 & \text{for } i \in I^\ast((p, q), (x, v)) 
\end{cases}
\quad g^y_i(s) = 0
\]

Rule 4: If for some \( i \in I \), \( s_j = ((p^j, q_j), (x_j, v^j)) \) where \( p^j = p \) and \( v^j = v \) for all \( j \neq i \), \( s_i = ((p^i, q_i), (x_i, v^i)) \) where \( p^i \neq p \) or \( v^i \neq v \), and \( i \in I^\ast((p, q), (x, v)) \), then
\[
g(s) = \begin{cases} 
(x_i, ((\Omega - \sum_{k=1}^{n} v^k - x_i)/(n - 1))_{j \neq i}, f(v^i)) & \text{if } (x_i, f(v^i)) \in \Lambda^F(((\Omega - \sum_{k=1}^{n} v^k - \sum_{j \neq i} x_j), x_{-i}), f(v)), \left(p, \frac{1}{1.MRT(v, f(v))} - \sum_{j \neq i} q_j, q_{-i}\right) \\
(((\Omega - \sum_{k=1}^{n} v^k - \sum_{j \neq i} x_j), x_{-i}), f(v)) & \text{otherwise}
\end{cases}
\]

32
Rule 5: For any other case, \( g_i(s) = (\Omega - \sum_{k=1}^m v^{i \cdot k}), f(v^i) \) and \( g_j(s) = (0, f(v^j)) \) for all \( j \neq i^* \) where \( s_{i^*} = ((p_{i^*}, q_{i^*}), (x_{i^*}, v^{i^*})) \) and \( i^* \) is defined as follows. Let \( \sum_{i \in I} x_{i1} = \alpha \). Since \( x_i \in Q_x, x_{i1} \in [0, \Omega_1] \) for all \( i \). Let \( \beta + \gamma = \alpha \), where \( \beta \) is the largest integer less than or equal to \( \alpha \). Then for \( \gamma \in [0, 1) \), there is a unique \( i^* \in I \) such that \( \gamma \in [(i^* - 1)/n, i^*/n) \).

The remain of the proof is analogous with the case where \( l = 1 \) and \( m = 1 \).

### Proof of Proposition 5.2

By Proposition 3.3, the proof of necessity is analogous with the case where \( l = 1 \) and \( m = 1 \).

Next, we prove sufficiency by constructing a mechanism.

**Rule 1:** If \( s \) is \( P^2Q \)-consistent with \((p, q) \) and \( v, (p, q) \in \Psi(f(v), v, T), I((p, q), (x, v)) = I, \) and \((x, f(v)) \in A, \) then \( g(s) = (x, f(v)) \).

**Rule 2:** If \( s \) is \( P^2Q \)-consistent with \((p, q) \) and \( v, (p, q) \in \Psi(f(v), v, T), I((p, q), (x, v)) = I, \) and \((x, f(v)) \notin A, \) then \( g(s) = z((p, q), (x, v)) \).

**Rule 3:** If \( s \) is \( P^2Q \)-consistent with \((p, q) \) and \( v, (p, q) \in \Psi(f(v), v, T) \) and \( 1 \leq \sharp I((p, q), (x, v)) \leq n - 1, \) then
\[
\hat{g}_i^*(s) = \begin{cases} \Omega/(n - \sharp I((p, q), (x, v))) & \text{for } i \notin I((p, q), (x, v)) \, \text{for } i \in I((p, q), (x, v)), \quad g_i^p(s) = 0 \end{cases}
\]

**Rule 4:** If for some \( i \in I, s_{-i} \) is \( P^2Q \)-consistent with \((p, q) \) and \( v \) where \( (p, q) \in \Psi(f(v), v, T), s_i = (p^i, q^i, q^i_{i+1}, (x_i, v^i)) \) where \( p^i \neq p, q^i_i \neq q^i_{i+1}, q^i_{i+1} \neq q^i_{i+2} \) or \( v^i \neq v, \) and \( i \in I((p, q), (x, v)) \), then
\[
g(s) = \begin{cases} (x_i, ((\Omega - \sum_{k=1}^m v^i - x_i)/(n - 1))_{j \neq i}, f(v^i)), & \text{if } (x_i, f(v^i)) \in A_i^e((\Omega - \sum_{k=1}^m v^i - \sum_{j \neq i} x_j, x_{-i}), f(v)), (p, q)) \\ ((\Omega - \sum_{k=1}^m v^i - \sum_{j \neq i} x_j, x_{-i}), f(v)) & \text{otherwise} \end{cases}
\]

**Rule 5:** For any other case, \( g_i(s) = (\Omega - \sum_{k=1}^m v^{i \cdot k}), f(v^i) \) and \( g_j(s) = (0, f(v^j)) \) for all \( j \neq i^* \) where \( s_{i^*} = ((p_{i^*}, q_{i^*}), (x_{i^*}, v^{i^*})) \) and \( i^* \) is defined as follows. Let \( \sum_{i \in I} x_{i1} = \alpha \). Since \( x_i \in Q_x, x_{i1} \in [0, \Omega_1] \) for all \( i \). Let \( \beta + \gamma = \alpha \), where \( \beta \) is the largest integer less than or equal to \( \alpha \). Then for \( \gamma \in [0, 1) \), there is a unique \( i^* \in I \) such that \( \gamma \in [(i^* - 1)/n, i^*/n) \).

The remain of the proof is analogous with the case where \( l = 1 \) and \( m = 1 \).

### Proof of Proposition 5.3

By Proposition 3.4, the proof of necessity is analogous with the case where \( l = 1 \) and \( m = 1 \).

Next, we prove sufficiency by constructing a mechanism.

**Rule 1:** If for all \( i \in I, s_i = ((p, q), (x_i, v)), (p, q) \in \Psi(f(v), v, T), I((p, q), (x, v)) = I, \) and \((x, f(v)) \in A, \) then \( g(s) = (x, f(v)) \).

**Rule 2:** If for all \( i \in I, s_i = ((p, q), (x_i, v)), (p, q) \in \Psi(f(v), v, T), I((p, q), (x, v)) = I, \) and \((x, f(v)) \notin A, \) then \( g(s) = z((p, q), (x, v)) \).

**Rule 3:** If for all \( i \in I, s_i = ((p, q), (x_i, v)), (p, q) \in \Psi(f(v), v, T) \) and \( 1 \leq \sharp I((p, q), (x, v)) \leq n - 1, \) then
\[
\hat{g}_i^*(s) = \begin{cases} \Omega/(n - \sharp I((p, q), (x, v))) & \text{for } i \notin I((p, q), (x, v)) \, \text{for } i \in I((p, q), (x, v)), \quad g_i^p(s) = 0 \end{cases}
\]

**Rule 4:** If for some \( i \in I, s_j = ((p, q)^j, (x_j, v^j)) \) where \((p, q)^j = (p, q), v^j = v \) and \( (p, q) \in \Psi(f(v), v, T) \) for all \( j \neq i, \)
\[ s_i = ((p,q)^i,(x_i,v^i)) \] where \((p,q)^i \neq (p,q)\) or \(v^i \neq v\), and \(i \in I((p,q),(x,v))\), then

\[
g(s) = \begin{cases} 
(x_i,((\Omega - \sum_{k=1}^{m} v^i_k - x_i)/(n - 1)), f(v^i)) & \text{if } (x_i, f(v^i)) \in \Lambda_i^f (((\Omega - \sum_{k=1}^{m} v^k - \sum_{j \neq i} x_j, x_{-i}), f(v)), (p,q)) \\
((\Omega - \sum_{k=1}^{m} v^k - \sum_{j \neq i} x_j, x_{-i}), f(v)) & \text{otherwise}
\end{cases}
\]

**Rule 5:** For any other case, \(g_i(s) = (\Omega - \sum_{k=1}^{m} v^i_k, f(v^i))\) and \(g_j(s) = (0, f(v^i))\) for all \(j \neq i^*\) where \(s_{i^*} = ((p,q)^{i^*},(x_{i^*},v^{i^*}))\) and \(i^*\) is defined as follows. Let \(\sum_{i \in I} x_{i1} = \alpha\). Since \(x_i \in Q_{x_i}, x_{i1} \in [0,\Omega_1]\) for all \(i\). Let \(\beta + \gamma = \alpha\), where \(\beta\) is the largest integer less than or equal to \(\alpha\). Then for \(\gamma \in [0,1)\), there is a unique \(i^* \in I\) such that \(\gamma \in [(i^* - 1)/n,i^*/n]\).

The remain of the proof is analogous with the case where \(l = 1\) and \(m = 1\).
$l = 1, m = 2, p = 1$

$(x, y):$ Fixed

Agent $i$ can manipulate $q_{i1}, q_{i2}$.

$\Rightarrow$ The personalized price vector of agent $i, q_i$, is not uniquely determined.

![Diagram](image1)

Figure 1. The personalized price of agent $i$ is not uniquely determined.

$l = 2, m = 1, p_1 = 1$

$(x, y):$ Fixed

Agent $i$ can not manipulate $p_2$, because a equilibrium price is common to other agents.

$\Rightarrow$ The personalized price of agent $i$, $q_i$, is uniquely determined.

![Diagram](image2)

Agent $i$’s budget constraint

Figure 2. The personalized price of agent $i$ is uniquely determined.
Figure 3. $P$ does not satisfy Condition $P^nQ$.

Kolm’s Triangle

$I(q, (x, y)) = I$

$s_i = ((x_i, y), q)$

$(x, y) \notin A$

$(\Omega - y - x_2, x_2) \in P(u)$

$\Lambda^c_i((\Omega - y - x_2, x_2), q)$

$u_2$

$u_1$

$u_2'$

$u_1'$

$(x_1, \Omega - y - x_1) \in P(u')$

$\Omega$

Figure 3. $P$ does not satisfy Condition $P^nQ$. 