Stable Coalition Structures under Restricted Coalitional Changes*

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Abstract

In this paper, we examine whether farsighted players form the efficient grand coalition structure in coalition formation games. We propose a stability concept for a coalition structure, called sequentially stability, when only bilateral mergers of two separate coalitions are feasible because of high negotiation costs. We provide an algorithm to check the sequential stability of the grand coalition structure as well as sufficient conditions for which the efficient grand coalition structure is sequentially stable. We also illustrate our results by means of common pool resource games and Cournot oligopoly games.

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1 Introduction

This paper examines the question of which coalition structures farsighted players form in coalition formation games. In economic environments with positive spillovers or externalities such as Cournot oligopolies, public goods economies, and common pool resource economies, the efficient grand coalition structure in which all players form one coalition is rarely a stable outcome in myopic notions of stability such as \( \alpha \)-stability, \( \beta \)-stability, \( \gamma \)-stability, and \( \delta \)-stability (see Hart and Kurz (1983) and Herings et al. (2010)). On the other hand, if coalition members are farsighted in that they consider the possibility that deviations might be countered by subsequent deviations, then the efficient grand coalition structure becomes stable in farsighted notions of stability in these economic games. Herings et al. (2010) showed that the singleton set consisting of only the efficient grand coalition structure is a von Neumann-Morgenstern farsightedly stable set introduced by Chwe (1994) and a farsightedly stable set due to Herings et al. (2010) in coalition formation games with positive spillovers. Moreover, Mauleon and Vannetelbosch (2004) established that the efficient grand coalition structure always belongs to the largest consistent set of Chwe (1994), and the largest cautious consistent set, a refinement of the largest consistent set they proposed, singles out the efficient grand coalition structure in those games.\(^1\)

In these previous studies on farsighted coalitional stability, no restriction is imposed on the feasible coalition structures that deviating coalitions can change. For instance, the inefficient singleton coalition structure in which no cooperation among players is formed can be directly changed into the efficient grand coalition structure in which all players cooperate. As Macho-Stadler et al. (2006) pointed out, however, such a merger may not be feasible when there are high negotiation or transaction costs with many players. Actually, mergers of more than two firms or organizations have been little often observed in comparison with bilateral mergers of two in many practical situations. This is because the costs of merging more than two organizations are much larger than that just between two.

For example, in Japan, all mergers of major banks after 1960 were bilateral and large banks have been formed through sequential processes of bilateral mergers. A typical example is the bank of Tokyo-Mitsubishi UFJ, Japan’s largest bank. It is the result of a merger between the bank of Tokyo-Mitsubishi, formerly Japan’s second-largest one, and the UFJ bank, which was Japan’s fourth-largest one in 2006. The bank of Tokyo-Mitsubishi, in turn, was the outcome of a merger of the Tokyo bank and the Mitsubishi bank in 1996, while the UFJ bank was the result of an integration between the Sanwa bank and the Tokai bank in 2002. See also Houston et al. (2001) and Macho-Stadler et al. (2006) for various examples of bilateral mergers in economic environments.

In this paper, we investigate the stability of coalition structures when only bilateral mergers of two separate coalitions are feasible because of high negotiation costs. More specifically, we consider the following definition of domination between two coalition structures with farsighted players. The coalition structure \( P_T \) is said

\(^1\)Efficient coalition structures may not be stable among farsighted players even when negotiation processes are unrestricted. See Diamantoudi and Xue (2007) for an example of such a game.
to \textit{sequentially dominate} the coalition structure $P_0$ if there is a sequence of coalition structures $\{P_t\}_{t=0}^T$ from $P_0$ to $P_T$ such that at each step $t$, one of the following holds: (1) Two separate coalitions in $P_t$ merge into one coalition in $P_{t+1}$ and no other change occurs. Each member in the two merging coalitions prefers his/her payoffs under the final coalition structure $P_T$ to that under the coalition structure $P_t$ before merging; or (2) One coalition in $P_t$ breaks up into two separate coalitions in $P_{t+1}$ and no other change occurs. Each member who leaves the coalition in $P_t$ prefers his/her payoff under the final coalition structure $P_T$ to that under the coalition structure $P_t$ before breaking up. A coalition structure is said to be \textit{sequentially stable} if it sequentially dominates all other coalition structures.

Notice that in (2) of the above definition of domination, we assume that only one breakup of a coalition into two separate coalitions happens at each negotiation process. This assumption regarding breaking-up of coalitions is a symmetric version of (1) concerning merging. This "step-by-step" approach, which allows only bilateral changes both in merging and in breaking-up at each step, is useful to describe negotiation steps concretely. One might say that a coalition could break up into any number of subcoalitions even with high negotiation costs. However, all of our results for the sequential stability turn out to hold even when any change of coalition structure is possible in breaking-up, whereas only bilateral changes are feasible in merging.

We provide an algorithm to check the sequential stability of the grand coalition structure which is applicable to any coalition formation game. We also give sufficient conditions for the grand coalition structure to be sequentially stable in coalition formation games with positive spillovers. As applications, we study common pool resource games and Cournot oligopoly games. We clarify how the sequential stability of the efficient grand coalition structure depends on the number of players as well as the production function of a common pool resource game. Contrary to the case of no restriction on feasible coalition deviations, the grand coalition structure may not be sequentially stable even among farsighted players when only bilateral mergers of two coalitions are possible. Nevertheless, for each possible number of players, there exists a class of concave production functions for which the grand coalition structure is sequentially stable. We also demonstrate that the efficient grand coalition structure may or may not be sequentially stable depending on the number of players in Cournot oligopoly games with linear demand.

Moreover, we examine what would happen if more than two coalitions could merge into one coalition because of smaller negotiation or transaction costs. We identify how many coalitions should merge into one coalition at each step of sequential domination to make the efficient grand coalition stable in common pool resource games and oligopoly games. We find that the merger of all singleton coalitions into the grand coalition is not necessary and a merger of a smaller number of coalitions is sufficient for the efficient grand coalition structure to be stable among farsighted players.

Macho-Stadler et al. (2006) also considered endogenous coalition formation only through bilateral mergers in the presence of high transaction costs in a Cournot oligopoly. They proposed a particular mechanism of sequential coalition formation and showed that the grand coalition structure is the subgame perfect equilibrium outcome of their mechanism if the number of firms is sufficiently large and the discount
factor is high enough. However, coalitions are allowed only to merge, but not to break up in their game. Since each coalition has a strong incentive to deviate from the efficient grand coalition in a Cournot oligopoly, it may be difficult to achieve efficiency if breakups of coalitions are possible.

The rest of the paper is organized as follows. In Section 2, we define the sequential stability of coalition structures and compare our notion with a von Neumann-Morgenstern farsightedly stable set, a farsightedly stable set, and the largest consistent set. Section 2 also provides an algorithm to check the sequential stability and sufficient conditions for the grand coalition structure to be sequentially stable. We examine the sequential stability of the efficient grand coalition structure in common pool resource games in Section 3 and in Cournot oligopoly games in Section 4. In Section 5, we extend our stability concept to the case in which more than two coalitions are allowed to merge into one coalition. Section 6 contains some concluding remarks.

2 Sequentially Stable Coalition Structures

2.1 Notation and Definitions

Let \( N = \{1, 2, ..., n\} \) be a set of players. A non-empty subset \( S \) of \( N \) is called a coalition. We use the concept of a coalition structure to express which coalitions players form: a coalition structure \( \mathcal{P} \) is a partition \( \{S_1, S_2, ..., S_k\} \) of \( N \), where \( S_j \neq \emptyset \) for \( i = 1, ..., k \), \( S_i \cap S_j = \emptyset \) for \( i \neq j \), and \( \bigcup_{j=1}^{k} S_j = N \). The set of partitions of \( N \) is denoted by \( \Pi \). In particular, \( \mathcal{P}^N = \{N\} \) is called the grand coalition structure, and \( \mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\} \) is said to be the singleton coalition structure. We assume that given any coalition structure \( \mathcal{P} \in \Pi \), the feasible payoff vector under \( \mathcal{P} \), \( u(\mathcal{P}) = (u_1(\mathcal{P}), u_2(\mathcal{P}), ..., u_n(\mathcal{P})) \in \mathbb{R}^n \), is uniquely determined. Here \( u_i(\mathcal{P}) \) denotes player \( i \)'s payoff when the coalition structure \( \mathcal{P} \) is formed. The triple \( (N, \Pi, (u_i)_{i \in N}) \) is called a coalition formation game.

We propose the following stability concept of a coalition structure called "sequential stability" for a coalition formation game among farsighted players. Let coalition structures \( \mathcal{P}, \mathcal{P}' \in \Pi \) with \( \mathcal{P} \neq \mathcal{P}' \) and a coalition \( Q \subseteq N \) with \( Q \neq \emptyset \) be given. We first define the obtainable coalition structures that a deviating coalition can enforce:

**Definition 1.** The coalition structure \( \mathcal{P}' \) is obtainable from \( \mathcal{P} \) via \( Q \) either by a merger of two coalitions or by a breakup into two coalitions if (i) \( \{S' \in \mathcal{P}' : S' \subseteq N \setminus Q\} = \{S \in \mathcal{P} : S \cap Q = \emptyset\} \cup \{S \setminus Q : S \in \mathcal{P}, S \cap Q \neq \emptyset\} \), and (ii) either (a) \( |\mathcal{P}'| = |\mathcal{P}| - 1, Q \in \mathcal{P}' \), and there are \( S_1, S_2 \in \mathcal{P} \) such that \( Q = S_1 \cup S_2 \), or (b) \( |\mathcal{P}'| = |\mathcal{P}| + 1, Q \in \mathcal{P}' \), and there are \( S \in \mathcal{P} \) and \( S' \in \mathcal{P}' \) such that \( Q = S \setminus S' \).

Condition (i) in Definition 1 says that only the players in \( Q \) deviate from their respective coalition(s) in \( \mathcal{P} \) and any non-deviating player in \( N \setminus Q \) does not move. Condition (ii) means that either (a) two separate coalitions \( S_1 \) and \( S_2 \) in \( \mathcal{P} \) merge into one coalition \( Q \) in \( \mathcal{P}' \), or (b) the players in \( Q \) leave their coalition \( S \) in \( \mathcal{P} \), and \( S \) breaks up into two separate coalitions \( S' \) and \( Q \) in \( \mathcal{P}' \). No other change occurs.

**Definition 2.** The coalition structure \( \mathcal{P} \) sequentially dominates \( \mathcal{P}' \) if there is a finite sequence of coalition structures \( \{\mathcal{P}_t\}_{t=0}^{T} \) such that
(1) $P_0 = P'$ and $P_T = P$,
(2) for any $t \in \{0, 1, ..., T - 1\}$, $P_{t+1}$ is obtainable from $P_t$ via some coalition $Q$ either by a merger of two coalitions or by a breakup into two coalitions, and $u_i(P_t) < u_i(P_T)$ for all $i \in Q$.

Condition (2) in Definition 2 specifies the restrictions on coalitional changes in the sequence of coalition structures $\{P_t\}_{t=0}^T$ and the requirements on the payoffs of the deviating coalitions. At each step $t$ in the sequence, one of the following should hold: (a) each member in the coalition $Q$ that is a merger of two coalitions in $P_t$ prefers his/her payoff under the final coalition structure $P_T$ to that under $P_t$ before merging; or (b) each member in $Q$ who leaves some coalition in $P_t$ prefers his/her payoff under the final coalition structure $P_T$ to that under $P_t$ before breaking-up.

**Definition 3.** The coalition structure $P$ is **sequentially stable** if for any other coalition structures $P' \neq P$, $P$ sequentially dominates $P'$.

### 2.2 Related Stability Concepts

Many previous studies on coalition formation games have investigated the case of no restriction on feasible coalitional changes:

**Definition 4.** The coalition structure $P'$ is **obtainable from** $P$ via $Q$ if (i) $\{S' \in P' : S' \subseteq N \setminus Q\} = \{S \in P : S \cap Q = \emptyset\} \cup \{S \setminus Q : S \in P, S \cap Q \neq \emptyset\}$, and (ii) there exist $S'_1, S'_2, ..., S'_m \in P'$ such that $\cup_{i=1}^m S'_i = Q$.

In this definition introduced by Herings et al. (2010), there is no restriction on the obtainable coalition structures that a deviating coalition can enforce, and all possibilities of refining and merging are allowed.

**Definition 5.** The coalition structure $P$ **indirectly dominates** $P'$ if there is a finite sequence of coalition structures $\{P_t\}_{t=0}^T$ such that

(1) $P_0 = P'$ and $P_T = P$,
(2) for any $t \in \{0, 1, ..., T - 1\}$, $P_{t+1}$ is obtainable from $P_t$ via some coalition $Q$ and $u_i(P_t) < u_i(P_T)$ for all $i \in Q$.\(^2\)

Chwe (1994) introduced the following stability concept by replacing the direct dominance relation with the indirect dominance relation of Harsanyi (1974) in the original definition of the stable set due to von Neumann and Morgenstern (1944):

**Definition 6.** The set $F \subseteq \Pi$ is a **von Neumann-Morgenstern farsightedly stable set** if

\(^2\)A weaker version of indirect dominance has been studied in the literature on farsighted coalitional stability: $u_i(P_t) < u_i(P_T)$ for some $i \in Q$ and $u_i(P_t) \leq u_i(P_T)$ for all $i \in Q$ (e.g., see Mauleon and Vannetelbosch (2004) and Herings et al. (2010)). In this definition, a coalition can deviate only if at least one of its members becomes better off, whereas all other members are at least as well off. A weak version of sequential dominance can be defined in a similar way. Our results in this paper hold for these weak versions of dominance, too.
(i) *internal stability:* for any coalition structure $\mathcal{P}' \in \mathcal{F}$, there is no $\mathcal{P} \in \mathcal{F}$ such that $\mathcal{P}$ indirectly dominates $\mathcal{P}'$; and
(ii) *external stability:* for any coalition structure $\mathcal{P}' \notin \mathcal{F}$, there exists $\mathcal{P} \in \mathcal{F}$ such that $\mathcal{P}$ indirectly dominates $\mathcal{P}'$.

The notion of von Neumann-Morgenstern farsightedly stable set coincides with the notion of *Extended EBA* (EEBA) due to Diamantoudi and Xue (2007) who extended the concept of *Equilibrium Binding Agreement* (EBA) introduced by Ray and Vohra (1997). Ray and Vohra (1997) assumed that coalitions can only break up into smaller sizes of coalitions, but cannot merge into larger sizes of coalitions, whereas Diamantoudi and Xue (2007) allowed for any coalitional deviations including breaking-up as well as merging.

Chwe (1994) proposed the largest consistent set by replacing the internal and external stability conditions of the von Neumann-Morgenstern farsightedly stable set by the conditions that internal and external deviations should be deterred:

**Definition 7.** The set $\mathcal{F} \subseteq \Pi$ is a *consistent set* if $\mathcal{P} \in \mathcal{F}$ if and only if for all $Q \subseteq N$, for all $\mathcal{P}' \in \Pi \setminus \mathcal{P}$ such that $\mathcal{P}'$ is obtainable from $\mathcal{P}$ via $Q$, there exists $\mathcal{P}'' \in \mathcal{F}$ such that either (a) $\mathcal{P}'' = \mathcal{P}'$ or (b) $\mathcal{P}''$ indirectly dominates $\mathcal{P}'$ and we do not have $u_i(\mathcal{P}) < u_i(\mathcal{P}'')$ for all $i \in Q$. The *largest consistent set* is the consistent set containing any consistent set.

Herings et al. (2010) introduced a stability notion to predict which coalition structures are going to emerge in the long run when players are farsighted:

**Definition 8.** The set $\mathcal{F} \subseteq \Pi$ is a *farsightedly stable set* if
(i) for any $\mathcal{P} \in \mathcal{F}$ and any $\mathcal{P}' \notin \mathcal{F}$ such that $\mathcal{P}'$ is obtainable from $\mathcal{P}$ via $Q$, there exists $\mathcal{P}'' \in \mathcal{F}$ such that $\mathcal{P}''$ indirectly dominates $\mathcal{P}'$ and we do not have $u_i(\mathcal{P}) < u_i(\mathcal{P}'')$ for all $i \in Q$;
(ii) *external stability:* for $\mathcal{P}' \notin \mathcal{F}$, there exists $\mathcal{P} \in \mathcal{F}$ such that $\mathcal{P}$ indirectly dominates $\mathcal{P}'$; and
(iii) there is no $\mathcal{F}' \subsetneq \mathcal{F}$ such that $\mathcal{F}'$ satisfies Conditions (i) and (ii).

Herings et al. (2010) showed that a farsightedly stable set always exists, whereas there may be no von Neumann-Morgenstern farsightedly stable set. Furthermore, they proved the following relationships:

**Proposition 1.** (Herings et al. (2010))
(i) If $\mathcal{F}$ is a von Neumann-Morgenstern farsightedly stable set, then it is a farsightedly stable set.
(ii) If $\{\mathcal{P}\}$ is a farsightedly stable set, then $\mathcal{P}$ belongs to the largest consistent set.

Suppose that a coalition structure $\mathcal{P}$ sequentially dominates any other coalition structure. Then the singleton set consisting of $\mathcal{P}$ satisfies the external stability condition in Definition 6 because sequential domination implies indirect domination. Also,
it is obvious that this singleton set satisfies the internal stability condition in Definition 6. Hence, by Proposition 1, we have the following relationships:

**Corollary 1.** Suppose that $P$ is sequentially stable. Then $\{P\}$ is a von Neumann-Morgenstern farsightedly stable set (an EEBA) and a farsightedly stable set. Moreover, $P$ belongs to the largest consistent set.\(^3\)

However, the converse is not true: the singleton set consisting of one coalition structure that is not sequentially stable may be a von Neumann-Morgenstern farsightedly stable set (an EEBA) and a farsightedly stable set, and the coalition structure may belong to the largest consistent set. In the next subsection, we will give an example illustrating this fact (Example 1) after describing an algorithm to check whether or not the grand coalition structure is sequentially stable.

### 2.3 An Algorithm to Check the Sequential Stability of the Grand Coalition Structure

In this subsection, we give an algorithm to check whether or not the grand coalition structure $P^N$ is sequentially stable. This algorithm can be applied to any coalition formation game.

Let $\Pi(k) \equiv \{P \in \Pi : |P| = k\}$ be the set of all coalition structures consisting of $k$ coalitions ($k = 1, 2, ..., n$) and $\Pi \equiv \cup_{k=1}^{n} \Pi(k)$ be the set of all possible coalition structures. Also, let $\Pi^*(k) \subseteq \Pi(k)$ be the set of coalition structures that are sequentially dominated by $P^N$, which can be found by using the following algorithm ($k = 2, 3, ..., n$).

**Step 1** (Finding $\Pi^*(2)$): Find coalition structures $P$ such that $|P| = 2$ and $u_i(P) < u_i(P^N)$ for all $i \in N$, and add all such coalition structures to $\Pi^*(2)$.

If $\Pi^*(2) = \emptyset$, then $P^N$ is not sequentially stable. (END)

Otherwise, go to Step 2.

**Step 2:**

Step 2-1 (Finding $\Pi^*(3)$):

a) Pick a coalition structure $P \in \Pi^*(2)$.

b) Find every coalition structure $P' \in \Pi(3)$ from which $P$ is obtainable via $S$ by a merger of two coalitions and $u_i(P') < u_i(P^N)$ for all $i \in S$.

Let $\Pi(3, P)$ be the set of all coalition structures found in Step b).

Repeat a) and b) for every $P \in \Pi^*(2)$. Let $\Pi^*(3) = \cup_{P \in \Pi^*(2)} \Pi(3, P)$.

If $\Pi^*(3) = \emptyset$, then $P^N$ is not sequentially stable. (END)

Step 2-2 (Expanding $\Pi^*(2)$ through $\Pi^*(3)$):

a) If $\Pi^*(2) = \Pi(2)$, skip Step 2-2, and go to Step 3. Otherwise, pick any coalition structure $P \in \Pi(2) \setminus \Pi^*(2)$.

b) Find a coalition structure $P' \in \Pi^*(3)$ which is obtainable from $P$ via $S$ by a breakup into two coalitions and $u_i(P) < u_i(P^N)$ for all $i \in S$.

\(^3\)We are grateful to an anonymous referee for pointing out this result.
If such a coalition structure $P'$ exists, add $P$ to $\Pi^*(2)$.

Repeat Step a) and Step b) until there is no such coalition structure $P'$ for any $P \in \Pi(2) \setminus \Pi^*(2)$.

Then go back to Step 2-1 again to find $\Pi^*(3)$ based on $P \in \Pi^*(2)$ which is newly added. Next go to Step 2-2 again to expand $\Pi^*(2)$. Repeat Steps 2-1 and 2-2 till there are no additional changes on $\Pi^*(3)$ and $\Pi^*(2)$. Then go to Step 3.

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**Step 3:**

Step 3-1 (Finding $\Pi^*(4)$):

a) Pick a coalition structure $P \in \Pi^*(3)$.

b) Find every coalition structure $P' \in \Pi(4)$ from which $P$ is obtainable via $S$ by a merger of two coalitions and $u_i(P') < u_i(P^N)$ for all $i \in S$.

Let $\Pi(4, P)$ be the set of all coalition structures found in Step b).

Repeat a) and b) for every $P \in \Pi^*(3)$. Let $\Pi^*(4) = \bigcup_{P \in \Pi^*(3)} \Pi(4, P)$.

If $\Pi^*(4) = \emptyset$, then $P^N$ is not sequentially stable. (END)

Step 3-2 (Expanding $\Pi^*(3)$ through $\Pi^*(4)$):

a) If $\Pi^*(3) = \Pi(3)$, skip Step 3-2, and go to Step 4. Otherwise, pick any coalition structure $P \in \Pi(3) \setminus \Pi^*(3)$.

b) Find a coalition structure $P' \in \Pi(4)$ which is obtainable from $P$ via $S$ by a breakup into two coalitions and $u_i(P) < u_i(P^N)$ for all $i \in S$.

If such a coalition structure $P'$ exists, add $P$ to $\Pi^*(3)$.

Repeat Step a) and Step b) until there is no such coalition structure $P'$ for any $P \in \Pi(3) \setminus \Pi^*(3)$.

Then go back to Step 3-1 again to find $\Pi^*(4)$ based on $\Pi^*(3)$. Next go to Step 3-2 again to expand $\Pi^*(2)$. Repeat Steps 3-1 and 3-2 till there are no additional changes on $\Pi^*(4)$ and $\Pi^*(3)$. Then go to Step 3-3.

Step 3-3 (Expanding $\Pi^*(2)$ through $\Pi^*(3)$):

a) If $\Pi^*(2) = \Pi(2)$, skip this step, and go to Step 3-5. Otherwise, pick any coalition structure $P \in \Pi(2) \setminus \Pi^*(2)$.

b) Find a coalition structure $P' \in \Pi^*(3)$ which is obtainable from $P$ via $S$ by a breakup into two coalitions and $u_i(P) < u_i(P^N)$ for all $i \in S$.

Repeat Step a) and Step b) until there is no such coalition structure $P'$ for any $P \in \Pi(2) \setminus \Pi^*(2)$. Then go to Step 3-4.

Step 3-4 (Expanding $\Pi^*(3)$ through $\Pi^*(2)$, the same as Step 2-1):

a) Pick a coalition structure $P \in \Pi^*(2)$.

b) Find every coalition structure $P' \in \Pi(3)$ from which $P$ is obtainable via $S$ by a merger of two coalitions and $u_i(P') < u_i(P^N)$ for all $i \in S$.

Let $\Pi(3, P)$ be the set of all coalition structures found in Step b).

Repeat a) and b) for every $P \in \Pi^*(2)$. Let $\Pi^*(3) = \bigcup_{P \in \Pi^*(2)} \Pi(3, P)$.

Repeat Step a) and Step b) until there is no such coalition structure $P'$ for any $P \in \Pi(2) \setminus \Pi^*(2)$. 

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Then go back to Step 3-3 again to find $\Pi^*(2)$ based on $\Pi^*(3)$. Next go to Step 3-4 again to expand $\Pi^*(3)$. Repeat Steps 3-3 and 3-4 till there are no additional changes on $\Pi^*(2)$ and $\Pi^*(3)$.

Then repeat Steps 3-1, 3-2, 3-3, 3-4 (Expanding $\Pi^*(4)$, $\Pi^*(3)$, $\Pi^*(2)$). If there are no additional changes in $\Pi^*(4)$, $\Pi^*(3)$, and $\Pi^*(2)$, then go to Step 4.

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**Step 4:**
Step 4-1 (Finding $\Pi^*(5)$ based on $\Pi^*(4)$).
Step 4-2 (Expanding $\Pi^*(4)$ through $\Pi^*(5)$), and repetitions of Steps 4-1 and 4-2.
Step 4-3 (Expanding $\Pi^*(3)$ through $\Pi^*(4)$)
Step 4-4 (Expanding $\Pi^*(4)$ through $\Pi^*(3)$), and repetitions of Steps 4-3 and 4-4.
Step 4-5 (Expanding $\Pi^*(2)$ through $\Pi^*(3)$)
Step 4-6 (Expanding $\Pi^*(3)$ through $\Pi^*(2)$) and repetitions of Steps 4-5 and 4-6.
Step 4-7 (Expanding $\Pi^*(4)$ through $\Pi^*(3)$)
Step 4-8 (Expanding $\Pi^*(3)$ through $\Pi^*(4)$) and repetitions of Steps 4-7 and 4-8.

Repeat Steps 4-1, 4-2, 4-3, 4-4, 4-5, 4-6, 4-7, and 4-8. If there are no additional changes in $\Pi^*(5)$, $\Pi^*(4)$, $\Pi^*(3)$, and $\Pi^*(2)$, then go to Step 5.

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After **Step 5**, **Step 6**, ..., **Step n**, if $\Pi^*(2) = \Pi(2), \Pi^*(3) = \Pi(3), ..., \Pi^*(n) = \Pi(n)$, then $\mathcal{P}^N$ is sequentially stable. Otherwise, $\mathcal{P}^N$ is not sequentially stable. (END)

Since this algorithm covers all the possibilities of sequential dominance by the grand coalition structure, we can determine whether or not the grand coalition structure is sequentially stable.

In this algorithm, once $\Pi^*(k) = \Pi(k)$ holds for some $k = 2, ..., n$, the expanding steps of $\Pi^*(k)$ can be omitted after that.

Moreover if we could find $\Pi^*(k) = \Pi(k)$ for some $k = 2, ..., n$ by some way, we can guarantee the sequential stability of the grand coalition structure by the following simpler check:

Suppose there exists $k$ $(2 \leq k \leq n)$ which satisfies

(1) $\Pi(k) = \Pi^*(k)$;

(2) for any $\mathcal{P} \in \Pi(l), 2 \leq l < k$, there is $S \in \mathcal{P}$ such that $|S| \geq 2$ and $u_i(\mathcal{P}) < u_i(\mathcal{P}^N)$ for some $i \in S$; and

(3) for any $\mathcal{P} \in \Pi(l), k < l \leq n$, for any $S \in \mathcal{P}$, $u_i(\mathcal{P}) < u_i(\mathcal{P}^N)$ for all $i \in S$.

For this case, we can go to **Step k** directly and skip the other steps, **Step l** $(2 \leq l < k)$. Then through the sub-steps of Step k, the condition (2) implies that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$ for $2 \leq |\mathcal{P}| < k$. Moreover through Steps $l - 1, l = k + 1, k + 2, ..., n$, the condition (3) implies that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$ for $k < |\mathcal{P}| \leq n$. This shows the sequential stability of the grand coalition structure.

As an illustration of the above algorithm, let us examine the following example.

**Example 1.** Consider a five-person game where payoffs are obtained from a model of an economy with a common pool resource that we will investigate in more detail in Section 3: $u_i(\mathcal{P}) = v(\mathcal{P})/|S|$ for all $S \in \mathcal{P}$ and all $i \in S$, where $v(\mathcal{P}^N) = 10,
\( v(\mathcal{P}_2) = 18 \) for all \( \mathcal{P}_2 \) with \( |\mathcal{P}_2| = 2 \), \( v(\mathcal{P}_3) = 8 \) for all \( \mathcal{P}_3 \) with \( |\mathcal{P}_3| = 3 \), \( v(\mathcal{P}_4) = 5 \) for all \( \mathcal{P}_4 \) with \( |\mathcal{P}_4| = 4 \), and \( v(\mathcal{P}_5) = 3 \). Figure 1 illustrates every possible coalition structure in which the circle means a coalition and the number in the circle indicates the cardinality of the coalition. The vector under each coalition \( S \) indicates their payoffs, \( (u_i(\mathcal{P}))_{i \in S} \).

We show that the grand coalition structure \( \mathcal{P}^N = \{5\} \) is sequentially stable by applying our algorithm. Let us denote a coalition structure \( \mathcal{P} = \{S_1, S_2, S_3, \ldots, S_k\} \), where \( |S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq \ldots \leq |S_k| = r_k \), by \( \{r_1; r_2; r_3; \ldots; r_k\} \).

We also indicate a sequence of coalition structures for which \( \mathcal{P} = \mathcal{P}_T \) sequentially dominates \( \mathcal{P}' = \mathcal{P}_0 \) by \( \mathcal{P}_0 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_2 \rightarrow \ldots \rightarrow \mathcal{P}_T \).

**Step 1.** Finding \( \Pi^*(2) \): \( \{5\} \) is obtainable from \( \{2; 3\} \) when the two-person coalition and the three-person coalition merge into the grand coalition, and \( u_i(\{5\}) > u_i(\{2; 3\}) \) for every player \( i \). Therefore, \( \{2; 3\} \rightarrow \{5\} \) and \( \{2; 3\} \in \Pi^*(2) \).

**Step 2-1(A).** Finding \( \Pi^*(3) \): \( \{2; 3\} \in \Pi^*(2) \) is obtainable from \( \{1; 1; 3\} \) when two singleton coalitions merge into the two-person coalition, and \( u_i(\{5\}) > u_i(\{1; 1; 3\}) \) for every player \( i \). By this fact and Step 1, \( \{1; 1; 3\} \rightarrow \{2; 3\} \rightarrow \{5\} \) and \( \{1; 1; 3\} \in \Pi^*(3) \).

**Step 2-2(B).** Finding \( \Pi^*(3) \): \( \{2; 3\} \in \Pi^*(2) \) is obtainable from \( \{1; 2; 2\} \) when the singleton coalition and one of the two-person coalitions merge into the three-person coalition, and \( u_i(\{5\}) > u_i(\{1; 2; 2\}) \) for every player \( i \). By this fact and Step 1, \( \{1; 2; 2\} \rightarrow \{2; 3\} \rightarrow \{5\} \) and \( \{1; 2; 2\} \in \Pi^*(3) \).

**Step 2-2.** Expanding \( \Pi^*(2) \) through \( \Pi^*(3) \): \( \{1; 1; 3\} \in \Pi^*(3) \) is obtainable from \( \{1; 4\} \) by the deviation of one person from the 4-person coalition, and \( u_i(\{5\}) > u_i(\{1; 4\}) \) for each player \( i \) in the 4-person coalition. By this fact and Step 2-1(A), \( \{1; 4\} \rightarrow \{1; 1; 3\} \rightarrow \{2; 3\} \rightarrow \{5\} \) and \( \{1; 1; 3\} \in \Pi^*(2) \).

**Step 3.** Finding \( \Pi^*(4) \): \( \{1; 1; 3\} \in \Pi^*(3) \) is obtainable from \( \{1; 1; 1; 2\} \) when one singleton coalition and the two-person coalition merge into the three-person coalition, and \( u_i(\{5\}) > u_i(\{1; 1; 1; 2\}) \) for every player \( i \). By this fact and Step 2-1(A), \( \{1; 1; 1; 2\} \rightarrow \{1; 1; 3\} \rightarrow \{2; 3\} \rightarrow \{5\} \), and \( \{1; 1; 1; 2\} \in \Pi^*(4) \).

**Step 4.** Finding \( \Pi^*(5) \): \( \{1; 1; 1; 2\} \in \Pi^*(4) \) is obtainable from \( \{1; 1; 1; 1\} \) when two singleton coalitions merge into the two-person coalition, and \( u_i(\{5\}) > u_i(\{1; 1; 1; 1\}) \) for every player \( i \). By this fact and Step 3, \( \{1; 1; 1; 1\} \rightarrow \{1; 1; 1; 2\} \rightarrow \{1; 1; 3\} \rightarrow \{2; 3\} \rightarrow \{5\} \) and \( \{1; 1; 1; 1\} \in \Pi^*(5) \).

Now we have \( \Pi^*(k) = \Pi(k) \) for all \( k = 2, 3, 4, 5 \), that is, \( \mathcal{P}^N = \{5\} \) sequentially dominates any other coalition structure.

Moreover, \( \mathcal{P}^N \) is the unique sequentially stable coalition structure. The reason is as follows. First of all, any coalition structure \( \mathcal{P} \) except for \( \mathcal{P}^N \) and \( \{1; 4\} \) is not sequentially stable because \( u_i(\mathcal{P}^N) > u_i(\mathcal{P}) \) for all \( i \in N \), so that \( \mathcal{P} \) cannot sequentially dominate \( \mathcal{P}^N \). Second, \( \{1; 4\} \) cannot sequentially dominate \( \{1; 1; 1; 2\} \).

If \( \{1; 4\} \) sequentially dominates \( \{1; 1; 1; 2\} \), then two coalitions including at least one singleton coalition in \( \{1; 1; 1; 2\} \) should merge at the first step. However, no such merger is profitable for any player in the singleton coalition because his/her payoff under \( \{1; 1; 1; 2\} \) is larger than that in the 4-person coalition under \( \{1; 4\} \). Therefore, \( \{1; 4\} \) is not sequentially stable.

On the other hand, \( \{1; 4\} \) indirectly dominates \( \{1; 1; 1; 2\} \) because the sequence \( \{1; 1; 1; 2\} \rightarrow \{1; 1; 1; 1\} \rightarrow \{1; 4\} \) satisfies the requirements of indirect dominance.
under no restriction on coalitional changes in Definition 5. Moreover, it is easy to check that \(\{1; 4\}\) indirectly dominates any other coalition structure. Hence, even though \(\{1; 4\}\) is not sequentially stable, the singleton set \(\{\{1; 4\}\}\) is a von Neumann-Morgenstern farsightedly stable set (an EEBA) as well as a farsightedly stable set, and \(\{1; 4\}\) belongs to the largest consistent set by Proposition 1. That is, the converse of Corollary 1 is not true.

2.4 Sufficient Conditions for the Sequential Stability of the Grand Coalition Structure

In the subsequent sections, we examine a class of coalition formation games satisfying the following conditions:

**Definition 9.** Equal sharing: for any \(\mathcal{P} \in \Pi\), any \(S \in \mathcal{P}\), and any \(i, j \in S\), \(u_i(\mathcal{P}) = u_j(\mathcal{P})\).

**Definition 10.** Negative Association: for any \(\mathcal{P} \in \Pi\) and any \(\mathcal{S}, \mathcal{T} \in \mathcal{P}\), \(|\mathcal{S}| < |\mathcal{T}|\) if and only if \(u_i(\mathcal{P}) > u_j(\mathcal{P})\) for any \(i \in \mathcal{S}\) and any \(j \in \mathcal{T}\).

**Definition 11.** Positive Spillovers: for any \(\mathcal{P}, \mathcal{P}' \in \Pi\) with \(|\mathcal{P}| < |\mathcal{P}'|\), any \(S \in \mathcal{P} \cap \mathcal{P}'\), and any \(i \in S\), \(u_i(\mathcal{P}) > u_i(\mathcal{P}')\).

**Definition 12.** Efficiency: for any \(\mathcal{P} \in \Pi\) with \(\mathcal{P} \neq \mathcal{P}^N\), \(\sum_{i=1}^{n} u_i(\mathcal{P}) < \sum_{i=1}^{n} u_i(\mathcal{P}^N)\).

Equal sharing means that the value of any coalition in any coalition structure is shared equally, implying that players are symmetric or identical. Negative association says that the per-member payoff becomes lower as the size of a coalition grows larger in any coalition structure. Positive spillovers implies that the formation of a coalition by other players increases the payoff of a player. Efficiency means that the grand coalition structure is the only efficient coalition structure maximizing the sum of payoffs over all players.

Coalition formation games in economic situations satisfying these four conditions include cartel formation games in Cournot oligopolies (e.g., Bloch (1996) and Yi (1997)), public good provision games (e.g., Yi (1997)), and common pool resource games (e.g., Funaki and Yamato (1999)).

Herings et al. (2010) showed that under the four conditions, the grand coalition structure \(\mathcal{P}^N\) indirectly dominates any other coalition structure under no restriction on coalitional changes, implying that the singleton set consisting of \(\mathcal{P}^N\) is a von Neumann-Morgenstern farsightedly stable set (an EEBA). The set \(\{\mathcal{P}^N\}\) is also farsightedly stable and \(\mathcal{P}^N\) belongs to the largest consistent set by Proposition 1.

Mauleon and Vannetelbosch (2004) studied a class of games satisfying the above four conditions as well as the individual free-riding condition: for any \(\mathcal{P} \in \Pi\), any \(S \in \mathcal{P}\), and any \(i \in S\), \(u_i(\mathcal{P}\{S\} \cup \{S\backslash \{i\}, \{i\}\}) > u_i(\mathcal{P})\). They proved that if for each non-symmetric coalition structure \(\mathcal{P} \in \Pi\) with \(\mathcal{P} \neq \mathcal{P}^N, \mathcal{P}^I\), there exists \(S \in \mathcal{P}\) such that \(u_i(\mathcal{P}^I) > u_i(\mathcal{P})\) for any \(i \in S\), then the grand coalition structure \(\mathcal{P}^N\) is the unique coalition structure in the largest consistent set. Mauleon and Vannetelbosch (2004)
also proposed a refinement of the largest consistent set called the largest cautious consistent set based on the assumption of players being cautious:

**Definition 13.** Let $Z^0 \equiv \Pi$. Then $Z^k$ ($k = 1, 2, \ldots$) is inductively defined as follows: $\mathcal{P} \in Z^{k-1}$ belongs to $Z^k$ if and only if for all $\mathcal{P}' \in \Pi \setminus \mathcal{P}$ such that $\mathcal{P}'$ is obtainable from $\mathcal{P}$ via $Q$, there exists $\alpha = (\alpha(\mathcal{P}^1), \ldots, \alpha(\mathcal{P}^m))$ satisfying $\sum_{j=1}^{m} \alpha(\mathcal{P}^j) = 1$, $\alpha(\mathcal{P}^j) \in (0, 1)$, that gives only positive weight to each $\mathcal{P}^j \in Z^{k-1}$, where $\mathcal{P}^j = \mathcal{P}'$ or $\mathcal{P}^j$ indirectly dominates $\mathcal{P}'$, such that we do not have

$$u_i(\mathcal{P}) < \sum_{\mathcal{P}^j \in Z^{k-1}, \mathcal{P}^j = \mathcal{P}' \text{ or } \mathcal{P}^j \text{ indirectly dominates } \mathcal{P}'} \alpha(\mathcal{P}^j) \cdot u_i(\mathcal{P}^j), \forall i \in Q.$$ 

The largest cautious consistent set is $\bigcap_{k \geq 1} Z^k$.

Mauleon and Vannetelbosch (2004) showed that the largest cautious consistent set singles out the efficient grand coalition structure $\mathcal{P}^N$. However, $\mathcal{P}^N$ may not be sequentially stable with restricted coalitional changes. We will show this fact for common pool resource games in Section 3 and for Cournot oligopoly games in Section 4.

In this subsection, we provide sufficient conditions for the grand coalition structure to be sequentially stable in coalition formation games.

**Theorem 1.** Consider any coalition formation game $(N, \Pi, (u_i)_{i \in N})$ satisfying equal sharing, negative association, positive spillovers, and efficiency. Let $n = 2^m + l$, where $m \geq 2$ and $0 \leq l \leq 2^m - 1$. If the inequalities

(a) $u_i(\mathcal{P}) < u_i(\mathcal{P}^N)$ for all $\mathcal{P}$ with $|\mathcal{P}| = 2^m - h - 1 + 2$ and for all $i \in S \in \mathcal{P}$ with $|S| = 2^h - 1$ ($h = 1, 2, \ldots, m - 1$) 

(b) $u_i(\mathcal{P}) < u_i(\mathcal{P}^N)$ for all $\mathcal{P}$ with $|\mathcal{P}| = 2$, and all $i \in S \in \mathcal{P}$ such that $|S| = 2^m - 1$

hold, then the grand coalition structure $\mathcal{P}^N$ is sequentially stable.4

The proof of Theorem 1 is given in Appendix. To see why the conditions in Theorem 1 are sufficient for the grand coalition structure $\mathcal{P}^N$ to be sequentially stable, let us consider the eight-person case as an example. The proof consists of 4 steps:

**Step 1:** The grand coalition structure $\mathcal{P}^N$ sequentially dominates some key coalition structure $\mathcal{P}^* = \{1; 1; 2; \ldots; 2; 2^{m-1} + l\}$ consisting of $2^{m-2} + 2$ coalitions, where $n = 2^m + l$.

For $n = 8$, $m = 3$, $l = 0$, and $\mathcal{P}^* = \{1; 1; 2; 4\}$. See Figure 2. Consider the sequence $\{1; 1; 2; 4\} \rightarrow \{2; 2; 4\} \rightarrow \{4; 4\} \rightarrow \{8\}$. At the first step of this sequence, two singleton coalitions merge into one 2-person coalition and this bilateral merger makes both players better off in the end of the sequence, $\{8\}$, by condition (a) in

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4In Theorem 1, efficiency can be replaced by the following condition of symmetry among coalitions: for any $\mathcal{P}, \mathcal{P}' \in \Pi$ with $|\mathcal{P}| = |\mathcal{P}'|$, any $S \in \mathcal{P}$, and any $S' \in \mathcal{P}'$, if $|S| = |S'|$, then $u_i(\mathcal{P}) = u_j(\mathcal{P}')$ for any $i \in S$ and any $j \in S'$. This condition says that under two coalition structures consisting of the same number of coalitions, all players who belong to coalitions comprising the same number of players receive the same payoff. Under this condition, equal sharing, negative association, and condition (a), the payoff of each person in any coalition of the maximal size under any coalition structure is smaller than that under the grand coalition structure. The proof is available upon request.
Theorem 1 for $m = 3$ and $h = 1$. Notice that $|\{1; 1; 2; 4\}| = 4$. At the second step, two 2-person coalitions merge into one 4-person coalition and they become better-off by this merger in the final coalition structure $\{8\}$ by condition (a) for $m = 3$ and $h = 2$. Notice that $|\{2; 2; 4\}| = 3$. At the third step, two 4-person coalitions merge into the grand coalition and this merger makes all players better off by condition (b) for $m = 3$. In this way, conditions (a) and (b) guarantee that $\mathcal{P}^N$ sequentially dominates $\mathcal{P}^*$.

**Step 2:** Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| = |\mathcal{P}^*|$ is sequentially dominated by $\mathcal{P}^N$.

For example, let us examine $\{1; 1; 1; 5\}$ consisting $|\mathcal{P}^*| = 4$ coalitions. Consider the sequence $\{1; 1; 1; 5\} \rightarrow \{1; 1; 1; 1; 4\} \rightarrow \{1; 1; 2; 4\}$. At the first step of this sequence, one person in the 5-person coalition of the maximal size deviates and forms a singleton coalition. It is easy to see from equal sharing, negative association, and efficiency that this person prefers his/her payoff in $\{8\}$ to that in $\{1; 1; 1; 5\}$. At the second step, two singletons merge into one coalition, and they prefer their payoffs in $\{8\}$ to that in $\{1; 1; 1; 1; 4\}$. The reason is as follows. Condition (a) for $m = 3$ and $h = 1$ implies that the payoff of every singleton player in any $\mathcal{P}$ with $|\mathcal{P}| = 4$ is smaller than that in $\{8\}$. This fact together with positive spillovers imply that every singleton player prefers his/her payoff in $\{8\}$ to that in any $\mathcal{P}'$ with $|\mathcal{P}'| > 4$. By combining the above sequence and the sequence in Step 1, we have the dominance sequence from $\{1; 1; 1; 5\}$ to $\{8\}$.

**Step 3:** Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| < |\mathcal{P}^*|$ other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$.

For instance, let us examine $\{1; 7\}$ consisting two coalitions. Consider the sequence $\{1; 7\} \rightarrow \{1; 1; 6\} \rightarrow \{1; 1; 5\}$. At the first step of this sequence, one person in the seven-person (six-person) coalition of the maximal size deviates and forms a singleton coalition, and this person prefers his/her payoff in $\{8\}$ to that in $\{1; 1; 7\}$. At the second step, two singletons merge into one coalition, and they prefer their payoffs in $\{8\}$ to that in $\{1; 1; 1; 5\}$. The reason is as follows. Condition (a) for $m = 3$ and $h = 1$ implies that the payoff of every singleton player in any $\mathcal{P}$ with $|\mathcal{P}| = 4$ is smaller than that in $\{8\}$. This fact together with positive spillovers imply that every singleton player prefers his/her payoff in $\{8\}$ to that in any $\mathcal{P}'$ with $|\mathcal{P}'| > 4$. By combining this sequence, the sequence in Step 2, and the sequence in Step 1, we have the dominance sequence from $\{1; 7\}$ to $\{8\}$.

**Step 4:** Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| > |\mathcal{P}^*| = 4$ is sequentially dominated by $\mathcal{P}^N$.

For example, let us examine $\{1; 1; 1; 1; 1; 1; 1; 1\}$ consisting eight singleton coalitions. Consider the sequence $\{1; 1; 1; 1; 1; 1; 1; 1\} \rightarrow \{1; 1; 1; 1; 1; 1; 2\} \rightarrow \{1; 1; 1; 1; 2; 2\} \rightarrow \{1; 1; 2; 2; 2\} \rightarrow \{1; 1; 2; 4\}$. At each step of this sequence, two coalitions merge into one coalition, and each player in the merged coalition prefer his/her payoff in $\{8\}$ to that before the merger. The reason is as follows. Condition (a) for $m = 3$ and $h = 1$ together with positive spillovers imply that the payoff of every singleton player under any coalition structure $\mathcal{P}$ containing more than $|\mathcal{P}^*| = 4$ coalitions is smaller than that under $\{8\}$, as shown in Step 2. This fact and negative association together yield that the payoff of every player in any coalition under $\mathcal{P}$ with $|\mathcal{P}^*| > 4$ is smaller than that under $\mathcal{P}^N$. By combining the above sequence and the sequence in Step 1, we have the dominance sequence from $\{1; 1; 1; 1; 1; 1; 1; 1\}$ to $\{8\}$. 

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In a similar way, we can construct a dominance sequence from any other coalition structure to the grand coalition structure via the key coalition structure \( \mathcal{P}^* = \{1; 1; 2; 4\} \), as illustrated in Figure 2.

We remark that the conditions (a) and (b) are only used to ensure that players become better off at the final grand coalition structure by mergers of two coalitions. Theorem 1 holds even when any change of coalition structure is possible in breaking-up, whereas only bilateral changes are feasible in merging.

We apply Theorem 1 to common pool resource games in the next section.

## 3 Common Pool Resource Games

Let us consider the following model of an economy with a common pool resource as examined by Weitzman (1974) and others. For each player \( i \in N \), let \( x_i \geq 0 \) represent the amount of labor input of \( i \). The overall amount of labor is given by \( \sum_{j \in N} x_j \).

The technology that determines the amount of product is considered to be a joint production function of the overall amount of labor \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying \( f(0) = 0 \), \( \lim_{x \to \infty} f'(x) = 0 \), \( f'(x) > 0 \), and \( f''(x) < 0 \) for \( x > 0 \). The distribution of the product is supposed to be proportional to the amount of labor expended by players. In other words, the amount of the product assigned to player \( i \) is given by \( \frac{x_i}{\sum_{j \in N} x_j} f(\sum_{j \in N} x_j) \).

The price of the product is normalized to be one unit of money and let \( q \) be a cost of labor per unit, and we suppose \( 0 < q < f'(0) \). Then individual \( i \)'s income is denoted by \( m_i(x_1, x_2, ..., x_n) = \frac{x_i}{x_S} f(x_N) - qx_i \). The total income of coalition \( S \) is given by \( m_S = \sum_{i \in S} m_i = \frac{x_S}{x_N} f(x_N) - qx_S \), where \( x_S = \sum_{i \in S} x_i \).

We consider a game where each coalition acts as a player. Each coalition chooses its total labor input and its payoff is given by the sum of the income over its members. The list \((x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_k})\) is a Nash equilibrium or simply an equilibrium under \( \mathcal{P} \) if \( m_{S_j}(x^*_{S_j}, x^*_{S_{-j}}) \geq m_{S_j}(x_{S_j}, x^*_{S_{-j}}) \) for all \( j \) and all \( x_{S_j} \in \mathbb{R}_+ \). It is not hard to check that there is a unique equilibrium under every coalition structure.\(^5\) Given a coalition structure \( \mathcal{P} = \{S_1, ..., S_k\} \), let \((x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))\) be a unique equilibrium under \( \mathcal{P} \) and let \( x^*_N(\mathcal{P}) = \sum_{i=1}^k x^*_{S_i}(\mathcal{P}) \). Moreover, let \( m^*_N(\mathcal{P}) = m_N(x^*_N(\mathcal{P}), ..., x^*_N(\mathcal{P})) \) be the equilibrium income of coalition \( S_i \) for \( i = 1, ..., k \) and therefore \( m^*_N(\mathcal{P}) = \sum_{i=1}^k m_{S_i}(x^*_{S_i}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P})) \). We assume that the payoff vector is given by \( u_i(\mathcal{P}) = \frac{m^*_N(\mathcal{P})}{|S_j|} \) for all \( i \in S_j \) and all \( S_j \in \mathcal{P} \).

We examine the sequential stability of the grand coalition structure \( \mathcal{P}^N \) in common pool resource games. First, let us consider the following example to illustrate the basic idea.

**Example 2.** Let \( n = 2, 3, ..., 8 \). The production function is given by \( f(x) = x^\alpha \), \( \alpha \in (0, 1) \). It is easy to see that for each \( \mathcal{P} = \{S_1, S_2, ..., S_k\} \) where \( k \) is a number of coalitions in \( \mathcal{P} \), the total amount of labor input is given by \( x^*_N(\mathcal{P}) = \left( \frac{\alpha-1+k}{kq} \right)^{1/(1-\alpha)} \) and the payoff of each player at equilibrium is provided by

\(^5\)See Theorem 1 in Funaki and Yamato (1999).
\[ u_i(P) = \frac{1-a}{|S_j|k^2} \left( \frac{a-1+k}{kq} \right)^{\alpha/(1-\alpha)}, \quad \forall i \in S_j, \forall j = 1, \ldots, k, \quad (5.1) \]

By using the algorithm in the previous section, we can check the sequential stability of the grand coalition structure \( P^N \). Whether \( P^N \) is sequentially stable depends on the number of players, \( n \), as well as on the value of production function parameter, \( \alpha \). Table 1 summarizes the result. Let \( \alpha(n) \) be \( \alpha \in (0, 1) \) such that \( \frac{1}{n} (\alpha)^{\alpha/(1-\alpha)} = \frac{1}{k^2} \left( \frac{a-1+k}{kq} \right)^{\alpha/(1-\alpha)} \), where \( k = 2 \) for \( n = 3 \), \( k = 3 \) for \( n = 5, 6, 7 \), and \( k = 4 \) for \( n = 8 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>The Sequential Stability of ( P^N )</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>Yes for all ( \alpha \in (0, 1) )</td>
</tr>
<tr>
<td>3</td>
<td>Yes if ( \alpha &lt; \alpha(3) \approx 0.2206 ) No, otherwise.</td>
</tr>
<tr>
<td>4</td>
<td>Yes for all ( \alpha \in (0, 1) )</td>
</tr>
<tr>
<td>5</td>
<td>Yes if ( \alpha &lt; \alpha(5) \approx 0.704302 ) No, otherwise.</td>
</tr>
<tr>
<td>6</td>
<td>Yes if ( \alpha &lt; \alpha(6) \approx 0.30501 ) No, otherwise.</td>
</tr>
<tr>
<td>7</td>
<td>Yes if ( \alpha &lt; \alpha(7) \approx 0.129913 ) No, otherwise.</td>
</tr>
<tr>
<td>8</td>
<td>Yes if ( \alpha &lt; \alpha(8) \approx 0.816629 ) No, otherwise.</td>
</tr>
</tbody>
</table>

Table 1. The Sequential Stability of \( P^N \) in Common Pool Resource Games.

For example, let \( n = 5 \) and \( \alpha(5) \) be \( \alpha \in (0, 1) \) satisfying \( \frac{1}{5} (\alpha)^{\alpha/(1-\alpha)} = \frac{1}{9} (\alpha + 2)^{\alpha/(1-\alpha)} \). Suppose that \( \alpha < \alpha(5) \approx 0.704302 \), implying that \( u_i(\{5\}) > \max_{i \in N} u_i(\{1;1;3\}) = \max_{i \in N} u_i(\{1;2;2\}) \) and \( u_i(\{5\}) > u_i(\{2;3\}) \) for \( i \in N \). Then we can show that \( P^N = \{5\} \) is sequentially stable by using the same argument as that in Example 1 in which the payoffs are based on the values of equation (5.1) for \( \alpha = 0.5 \) and \( q = 1/200 \). On the other hand, if \( \alpha \geq \alpha(5) \), then \( u_i(\{5\}) \leq \max_{i \in N} u_i(\{1;1;3\}) = \max_{i \in N} u_i(\{1;2;2\}) \) for \( i \in N \), implying that Step 2 in Example 1 does not hold, that is, \( \Pi^*(3) = \emptyset \). Hence, \( P^N = \{5\} \) is not sequentially stable.

In a similar way, we can prove the results described in Table 1 for \( n = 2, 3, 4, 6, 7, 8 \). The proof is available upon request.

As Table 1 illustrates, the efficient grand coalition structure \( P^N \) is sequentially stable for a sufficiently smaller value of \( \alpha \). As the marginal productivity increases, the gain by a deviation from \( P^N \) becomes greater, so that it is more difficult to achieve efficiency for a larger value of \( \alpha \in (0, 1) \).

Given any common pool resource game, we can check the sequential stability of the efficient grand coalition structure \( P^N \) by using the algorithm in Section 2, but it becomes more complicated as the number of players \( n \) increases. In order to obtain a general result for an arbitrary number of players, we apply Theorem 1 on sufficient conditions for the sequentially stability of \( P^N \) which are relatively easy to check. In particular, we find that when the production function is given by a power function, \( f(x) = x^\alpha \), for any number of players, there exists \( \alpha^* \in (0, 1) \) such that \( P^N \) is sequentially stable for \( \alpha < \alpha^* \).

Let a coalition structure \( P = \{S_1, S_2, \ldots, S_k\} \) with \(|P| = k\) be given. It is easy to see that for \( j = 1, \ldots, k \), the payoff of player \( i \in S_j \) is given by \( u_i(P) = m^2_{S_j}(P) / |S_j| = \).
Corollary 2. Let \( f(x_N^\ast(P)) - f'(x_N^\ast(P))x_N^\ast(P) \) if and only if \( B(k) < |S_j|/n \), where
\[
B(k) = \{ f(x_N^\ast(P)) - f'(x_N^\ast(P))x_N^\ast(P) \}/[k^2\{ f(x_N^\ast(P^N)) - f'(x_N^\ast(P^N))x_N^\ast(P^N) \}].
\]

By using this relation and Theorem 1, we obtain sufficient conditions of the sequential stability of the grand coalition structure in a common pool resource game:

**Theorem 2.** Let \( n = 2^m + l \), where \( m \geq 2 \) and \( 0 \leq l \leq 2^m - 1 \). If the inequalities
\[
B(2^m - h^2 - 2) < \frac{2^{m-1}}{n} \quad (h = 1, 2, \ldots, m - 1) \quad B(2) < \frac{2^{m-1}}{n}
\]
hold, then the grand coalition structure \( S^N \) is sequentially stable in a common pool resource game.

The proofs of all results in this section are in the appendix. If the production function is given by \( f(x) = x^\alpha \) (\( 0 < \alpha < 1 \)), then we have the following result:

**Theorem 3.** Consider a common pool resource game in which the production function is given by \( f(x) = x^\alpha \), \( \alpha \in (0, 1) \). For any number of players, \( n \), there exists \( \alpha^* \in (0, 1) \) such that the grand coalition structure \( S^N \) is sequentially stable for \( \alpha < \alpha^* \).

Theorem 3 says that for any number of players, there is a class of concave production functions for which the grand coalition structure is sequentially stable. In fact, as Table 1 illustrates, there may be a large region of \( \alpha \) for which \( S^N \) is sequentially stable.

Coalition structures other than the grand coalition structure could be sequentially stable in a common pool resource game. For example, in a 6-person game with \( f(x) = \sqrt{x} \), the coalition structures consisting of \((n - 1)\)-person coalition and one-person coalition, \( S^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\}\}(i \in N) \) are also sequentially stable. However, such a coalition structure is quite unfair in the sense that the payoff of the player in one-person coalition is equal to the sum of all other players’ payoffs. We examine under which condition these unfair coalition structures are not sequentially stable. For \( P \) with \( |P| = k \), let \( C(k) = \{ f(x_N^\ast(P)) - f'(x_N^\ast(P))x_N^\ast(P) \}/[f(x_N^\ast(P^N\setminus\{i\})) - f'(x_N^\ast(P^N\setminus\{i\}))x_N^\ast(P^N\setminus\{i\})] \}.

**Proposition 2.** Let \( n \geq 5 \). If \( C(3) \geq \frac{9}{8} \), then the coalition structures \( S^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\}\}(i \in N) \) are not sequentially stable in a common pool resource game.

By applying Proposition 2 to the case in which the production function is given by \( f(x) = x^\alpha \) (\( 0 < \alpha < 1 \)), we have the following:

**Corollary 2.** Let \( n \geq 5 \). If \( f(x) = x^\alpha \) and \( \alpha \geq 0.583804 \), then the coalition structures \( S^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\}\}, \{\{i\} \in N \) are not sequentially stable in a common pool resource game.

The above result shows that for any number of players, the unfair coalition structures \( S^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\} \) is not sequentially stable if \( \alpha \) is suitably large.\(^7\)

\(^6\)See Theorem 1 in Funaki and Yamato (1999).

\(^7\)The unfair coalition structure \( S^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\} \) may indirectly dominates any other coalition structure and hence the singleton set \( \{\{i\} \) may be a von Neumann-Morgenstern farsightedly
4 Cournot Oligopoly Games

In this section, we study the following Cournot oligopoly game with \( n \) identical firms producing a homogeneous good. Let \( x_i \) be firm \( i \)'s output \((i = 1, \ldots, n)\) and \( p \) be its price. The inverse demand function is linear: \( p = a - (\sum_{i=1}^{n} x_i) \), \( a > c > 0 \). The total cost function of firm \( i \) is \( cx_i \). Given a coalition structure \( \mathcal{P} = \{S_1, S_2, \ldots, S_k\} \), we assume that each coalition \( S_j \) is a player who chooses the total output level of its firms to maximize the sum of their profits, given the output levels of other coalitions. Also, every coalition is supposed to divide its profit equally among its members.

It is not hard to check that there exists a unique Nash equilibrium under each coalition structure. The total equilibrium output under a given coalition structure \( \mathcal{P} \) is given by \( x_N^{\mathcal{P}} = \frac{k(a-c)}{(k+1)} \) and the profit of each firm in coalition \( S_j \) belonging to \( \mathcal{P} \) are provided by \( u_i(\mathcal{P}) = \frac{((a-c)/(k+1))^2}{|S_j|} \) for each \( i \in S_j \) and each \( j \in \{1, \ldots, k\} \), where \( k = |\mathcal{P}| \). Without loss of generality, we assume that \( a - c = 1 \).

Mauleon and Vannetelbosch (2004) showed that many coalition structures including the efficient grand coalition structure \( \mathcal{P}^N \) may belong to the largest consistent set, whereas the largest cautious consistent set singles out \( \mathcal{P}^N \) under no restriction on possible deviations in Cournot oligopoly games. On the other hand, we find that \( \mathcal{P}^N \) is not sequentially stable for several cases when only bilateral mergers of two coalitions are possible. Table 2 summarizes the results regarding the sequential stability of \( \mathcal{P}^N \) for \( n \) less than or equal to 40. The proof is available upon request.

<table>
<thead>
<tr>
<th>( n )</th>
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<tr>
<td>( 3 \leq n \leq 14 )</td>
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<td>15</td>
<td>Yes</td>
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<tr>
<td>( 37 \leq n \leq 40 )</td>
<td>Yes</td>
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</tbody>
</table>

Table 2. The Sequential Stability of \( \mathcal{P}^N \) in a Cournot Oligopoly Game with Linear Demand.

Mauleon and Vannetelbosch (2004) showed that many coalition structures including the efficient grand coalition structure \( \mathcal{P}^N \) may belong to the largest consistent set, whereas the largest cautious consistent set singles out \( \mathcal{P}^N \) under no restriction on possible deviations in Cournot oligopoly games. On the other hand, we find that \( \mathcal{P}^N \) is not sequentially stable for several cases when only bilateral mergers of two coalitions are possible. Table 2 summarizes the results regarding the sequential stability of \( \mathcal{P}^N \) for \( n \) less than or equal to 40. The proof is available upon request.

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</tr>
<tr>
<td>( 37 \leq n \leq 40 )</td>
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</tbody>
</table>

Table 2. The Sequential Stability of \( \mathcal{P}^N \) in a Cournot Oligopoly Game with Linear Demand.

stable set (an EEBA) and a farsightedly stable set. It is difficult to eliminate this possibility because the singleton player gets the maximal payoff among the payoffs under all coalition structures, and the singleton coalition structure \( \mathcal{P}^I \) can merge into \( \mathcal{P}^{N\setminus\{i\}} \) directly at one step.
5 Extensions

The negative results in the previous sections regarding the sequential stability of the grand coalition structure could be overcome if the merger of all one-player coalitions into the grand coalition was allowed because of little negotiation or transaction costs. In fact, Herings et al. (2010) established that under the conditions of equal sharing, positive spillovers, negative association, and efficiency, the efficient grand coalition structure indirectly dominates any other coalition structure when this merger is possible.

However, we find that the merger of all singleton coalitions into the grand coalition is not necessary and a merger of a smaller number of coalitions is enough for the efficient grand coalition structure to be stable among farsighted players. In this section, we identify how many coalitions should merge into one coalition at each step of sequential domination to achieve efficiency in common pool resource games and Cournot oligopoly games. We introduce the following definitions which are generalizations of Definitions 1-3. Let coalition structures $\mathcal{P}, \mathcal{P}' \in \Pi$ with $\mathcal{P} \neq \mathcal{P}'$ and a coalition $Q \subseteq N$ with $Q \neq \emptyset$ be given.

**Definition 14.** Let $m$ and $b$ be integers between 2 and $n$. The coalition structure $\mathcal{P}'$ is obtained from $\mathcal{P}$ via $Q$ either by a merger of $m$ coalitions or by a breakup into $b$ coalitions if (i) $\{S' \in \mathcal{P}' : S' \subseteq N \setminus Q\} = \{S \in \mathcal{P} : S \cap Q = \emptyset\} \cup \{S \setminus Q : S \in \mathcal{P}, S \cap Q \neq \emptyset\}$, and (ii) either (a) $|\mathcal{P}'| = |\mathcal{P}| - (m - 1)$, $Q \in \mathcal{P}'$, and there are $S_1, S_2, ..., S_m \in \mathcal{P}$ such that $Q = S_1 \cup S_2 \cup ... \cup S_m$, or (b) $|\mathcal{P}'| = |\mathcal{P}| + b - 1$, and there are $S \in \mathcal{P}$ and $S_1, S_2, ..., S_b \in \mathcal{P}'$ such that $Q = S \setminus S_1$ and $Q = S_2 \cup S_3 \cup ... \cup S_b$.

Condition (ii) in Definition 14 means that either (a) $m$ separate coalitions, $S_1, S_2, ..., S_m$, in $\mathcal{P}$ merge into one coalition $Q$ in $\mathcal{P}'$, or (b) the players in $Q$ leave their coalition $S$ in $\mathcal{P}$, and $S$ breaks up into $b$ smaller coalitions $S_1, S_2, ..., S_b$ in $\mathcal{P}'$ and $Q = S_2 \cup S_3 \cup ... \cup S_b$. No other change occurs.

**Definition 15.** Let $\overline{m}$ and $\overline{b}$ be integers between 2 and $n$. We say that $\mathcal{P}$ sequentially dominates $\mathcal{P}'$ under mergers of at most $\overline{m}$ coalitions and breakups into at most $\overline{b}$ coalitions if there is a finite sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that

1. $\mathcal{P}_0 = \mathcal{P}'$ and $\mathcal{P}_T = \mathcal{P}$,
2. for any $t \in \{0, 1, ..., T - 1\}$, $\mathcal{P}_{t+1}$ is obtained from $\mathcal{P}_t$ via some coalition $Q$ either by a merger of $m$ coalitions or by a breakup into $b$ coalitions where $m \leq \overline{m}$ and $b \leq \overline{b}$, and $u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T)$ for all $i \in Q$.

**Definition 16.** Let $\overline{m}$ and $\overline{b}$ be integers between 2 and $n$. We say that the coalition structure $\mathcal{P}^* \in \Pi$ is $\overline{m} - \overline{b}$ sequentially stable if for all other coalition structures $\mathcal{P} \neq \mathcal{P}^*$, $\mathcal{P}^*$ sequentially dominates $\mathcal{P}$ under mergers of at most $\overline{m}$ coalitions and breakups into at most $\overline{b}$ coalitions.\(^8\)

We obtain the following result on this generalized stability concept for a common pool resource game:

\(^8\)Corollary 1 also holds for the $\overline{m} - \overline{b}$ sequential stability concept where $\overline{m}$ and $\overline{b}$ be any integers between 2 and $n$. 

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Theorem 4. Consider a common pool resource game in which the production function is given by \( f(x) = x^\alpha, \alpha \in (0, 1) \). Let \( n \geq 2 \) and \( b \) be any integer between 2 and \( n \). The grand coalition structure is \( m - b \) sequentially stable if \( m \) is the smallest integer satisfying \( m^2 \left( \frac{am}{n-m} \right)^{a/(1-a)} > n \).

The proofs of all results in this section are in the appendix. As the number of players \( n \) increases, \( m \) increases, but the ratio \( m/n \), that is, the relative size of mergers enough to achieve efficiency tends to become smaller. In addition, \( m \) becomes larger as \( \alpha \) increases, that is, the marginal productivity becomes larger. The following table illustrates these facts:

<table>
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<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>3</td>
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<td>2</td>
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<td>3</td>
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<td>11</td>
<td>23</td>
<td>33</td>
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</table>

Concerning a Cournot oligopoly game with linear demand, we have the following:

Theorem 5. Consider a Cournot oligopoly game in which the inverse demand function is given by \( p = a - \sum_{i=1}^{n} x_i \). Let \( n \geq 2 \) and \( b \) be any integer between 2 and \( n \). The grand coalition structure is \( m - b \) sequentially stable if \( m \) is the smallest integer satisfying \( m^2 \left( \frac{2m+1}{2m} \right)^2 > n \).

As the number of firms \( n \) increases, \( m \) increases, but the ratio \( m/n \), that is, the relative size of mergers enough to achieve efficiency tends to become smaller. The following table illustrates this fact:

<table>
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<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>( m )</td>
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<td>4</td>
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<td>12</td>
<td>14</td>
<td>19</td>
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<td>63</td>
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</table>

Notice that the condition in Theorem 5 is sufficient for the grand coalition to be sequentially stable, but not necessary. For example, if \( n = 40 \), then the grand coalition structure is \( 2 - 2 \) stable (see Table 2), but \( m = 12 \).

We also remark that the sufficient conditions for the generalized sequential stability in Theorems 4 and 5 depends only on \( m \), the maximal number of coalitions that can be merged, but not on \( b \), the maximal number of coalitions that can be broken up.

6 Concluding Remarks

In this paper we propose a sequentially stable coalition structure as a new concept of stability in coalition formation games if only bilateral mergers of two separate coalitions are feasible because of high negotiation costs. By using our algorithm to check the sequential stability of the grand coalition structure \( P^N \) and sufficient conditions for \( P^N \) to be sequentially stable, we find how the sequential stability of \( P^N \) depends on the number of players and the production function in common pool resource games and Cournot oligopoly games. We also identify how many coalitions
should merge into one coalition at each step of sequential domination to achieve efficiency in those games.

There are several open questions. First of all, we focus on checking the sequential stability of the grand coalition structure. It remains to investigate a condition for the existence of sequentially stable structures other than the grand coalition structure. In a Cournot oligopoly game with linear demand, no sequentially stable coalition structure may exist. Nevertheless, the efficient grand coalition becomes sequentially stable if mergers of many coalitions are allowed. It would be interesting to examine how large mergers should be allowed to achieve efficiency in other environments.

Moreover, the feasible payoff vector is assumed to be uniquely determined for each coalition structure in this paper. In a more general case, there could be a set of multiple feasible payoff vectors for a coalition structure. Then it is not easy to compare the present payoff to the final payoff because of the multiplicity of the feasible payoff vectors. We should take account of sequential domination between two feasible payoff vectors in the same coalition structure. This topic is also left for a future research.

7 Appendix

7.1 Proof of Theorem 1

In the following, we denote a coalition structure \( \mathcal{P} = \{S_1, S_2, S_3, ..., S_k\} \), where \(|S_i| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k\), by \( \{r_1; r_2; r_3; ...; r_k\} \), because payoffs are determined by the sizes of all coalitions in a coalition structure by equal sharing and negative association. Consider a coalition structure \( \mathcal{P}^* = \{1; 1; 2; 2; 2; 2; ...; 2; 2; 2m-1 + l\} \) consisting of \(2m-2\) coalitions. Let \(k^* = 2^{m-2} + 2\). We say that \(\mathcal{P}\) is a \(k\)-th stage coalition structure if \(|\mathcal{P}| = k\). The proof consists of four steps.

(Step 1) \(\mathcal{P}^*\) is sequentially dominated by \(\mathcal{P}^N\).

We have to find a sequence of coalition structures \(\{\mathcal{P}_t\}_{t=1}^{k^*}\) from \(\mathcal{P}_1 = \mathcal{P}^*\) to \(\mathcal{P}_{k^*} = \mathcal{P}^N\), where the two coalitions of the smallest size in \(\mathcal{P}_t\) merge into one coalition in \(\mathcal{P}_{t+1}\) for \(t = 1, 2, ..., 2m-2 + 1\). We will show the following is a domination sequence of coalition structures.

\[\mathcal{P}^* = \mathcal{P}_1 = \{1; 1; 2; 2; 2; 2; ...; 2; 2; 2m-1 + l\} \quad (2m-2+2)-th stage\]

\[\rightarrow \mathcal{P}_2 = \{2; 2; 2; 2; 2; 2; ...; 2; 2; 2m-1 + l\} \quad (2m-2+1)-th stage\]

\[\rightarrow \mathcal{P}_3 = \{4; 2; 2; 2; 2; ...; 2; 2; 2m-1 + l\} \quad (2m-2)-th stage\]

\[\rightarrow \mathcal{P}_4 = \{4; 4; 2; ...; 2; 2; 2m-1 + l\} \quad (2m-2-1)-th stage\]

\[\rightarrow ..., ..., \rightarrow \]

\[\rightarrow \mathcal{P}_{2m-3+1} = \{4; 4; 4; 4; ...; 2; 2; 2m-1 + l\} \quad (2m-3+2)-th stage\]

\[\rightarrow \mathcal{P}_{2m-3+2} = \{4; 4; 4; 4; ...; 4; 2m-1 + l\} \quad (2m-3+1)-th stage\]

\[\rightarrow \mathcal{P}_{2m-3+3} = \{8; 4; 4; ...; 4; 2m-1 + l\} \quad (2m-3)-th stage\]

\[\rightarrow ..., ..., \rightarrow \]
→ \mathcal{P}_{2m-3,2m-4+1} = \{8; 8; \ldots; 8; 4; 4; 2^{m-1} + l\} \ (2^{m-4} + 2)\text{-th stage})
→ \mathcal{P}_{2m-3,2m-4+2} = \{8; 8; \ldots; 8; 2^{m-1} + l\} \ (2^{m-4} + 1)\text{-th stage})
→ \ldots \to \ldots
→ \mathcal{P}_{(2m-3,2m-4,\ldots,2m-h)+2m-h-1+1} = \{2^h, 2^h, 2^h, \ldots, 2^h, 2^{h-1}, 2^{h-1}, 2^{m-1} + l\} \ (2^{m-h}+2)\text{-th stage})
→ \mathcal{P}_{(2m-3,2m-4,\ldots,2m-h)+2m-h-1+2} = \{2^h, 2^h, 2^h, \ldots, 2^h, 2^{m-1} + l\} \ (2^{m-h}^2+1)\text{-th stage})
→ \ldots \to \ldots
→ \mathcal{P}_{2m-2} = \{2^{m-3}, 2^{m-3}, 2^{m-3}, 2^{m-1} + l\} \ (5\text{-th stage})
→ \mathcal{P}_{2m-2} = \{2^{m-3}, 2^{m-3}, 2^{m-3}, 2^{m-1} + l\} \ (4\text{-th stage})
→ \mathcal{P}_{2m-2} = \{2^{m-2}, 2^{m-2}, 2^{m-1} + l\} \ (3\text{-th stage})
→ \mathcal{P}_{2m-2+1} = \{2^{m-1}, 2^{m-1} + l\} \ (2\text{-th stage}) \to \mathcal{P}_{2m-2+2} = \mathcal{P}^N

For \( t = 1 \), the payoff of every player in the two singletons under \( \mathcal{P}_1 \) is smaller than that under the final coalition structure \( \mathcal{P}^N \) by condition (a) for \( h = 1 \) in Theorem 1. Remark that \( |\mathcal{P}_1| = 2^{m-2} + 2 \).

For \( t = 2, 3, \ldots, 2^{m-3}+1 \), the payoff of every player in every 2-person coalition whose size is the minimal among the coalitions under \( \mathcal{P}_t \) is smaller than that under the final coalition structure \( \mathcal{P}^N \). Remark that \( |\mathcal{P}_{2m-3+1}| = 2^{m-3} + 2 < |\mathcal{P}_1| \) for all \( t \in \{2, 3, \ldots, 2^{m-3}\} \). Equal sharing, condition (a) for \( h = 2 \), and positive spillovers together imply that \( u_j(\mathcal{P}_t) = u_i(\mathcal{P}_t) > u_i(\mathcal{P}_{2m-3+1}) \) for all \( i \in S \) such that \( S \in \mathcal{P}_{2m-3+1} \) and \( |S| = 2 \), all \( j \in S \) such that \( S \in \mathcal{P}_t \) and \( |S| = 2 \), and all \( t \in \{2, 3, \ldots, 2^{m-3}\} \).

For \( t = 2^{m-3} + 2, 2^{m-3} + 3, \ldots, 2^{m-3} + 2^{m-4} + 1 \), the payoff of every player in 4-person coalitions under \( \mathcal{P}_t \) is smaller than that under the final coalition structure \( \mathcal{P}^N \). Remark that \( |\mathcal{P}_{2m-3+3,2m-4+1}| = 2^{m-4} + 2 < |\mathcal{P}_t| \) for all \( t \in \{2^{m-3} + 2, 2^{m-3} + 3, \ldots, 2^{m-3} + 2^{m-4} + 1\} \). Equal sharing, condition (a) for \( h = 3 \), and positive spillovers together imply that \( u_j(\mathcal{P}_t) = u_i(\mathcal{P}_t) > u_i(\mathcal{P}_{2m-3,2m-4+1}) \) for all \( i \in S \) such that \( S \in \mathcal{P}_{2m-3,2m-4+1} \) and \( |S| = 4 \), all \( j \in S \) such that \( S \in \mathcal{P}_t \) and \( |S| = 4 \), and all \( t \in \{2^{m-3} + 2, 2^{m-3} + 3, \ldots, 2^{m-3} + 2^{m-4}\} \).

Let \( h \in \{4, \ldots, m-3\} \). For \( t = (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 2, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 3, \ldots, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 2, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 3, \ldots, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 2, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 3, \ldots, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 4 \}\text{-th stage})
→ \ldots \to \ldots
→ \mathcal{P}_{2m-2} = \{2^{m-2}, 2^{m-2}, 2^{m-1} + l\} \ (3\text{-th stage})
→ \mathcal{P}_{2m-2+1} = \{2^{m-1}, 2^{m-1} + l\} \ (2\text{-th stage}) \to \mathcal{P}_{2m-2+2} = \mathcal{P}^N

For \( t = 2^{m-2} - 2 \) and \( 2^{m-2} - 1 \), which correspond to the case of \( h = m - 2 \) above, the payoff of every member in two \( 2^{m-3} \)-person coalitions under \( \mathcal{P}_{2m-2} \) and \( \mathcal{P}_{2m-2} \) is smaller than that under \( \mathcal{P}^N \). Remark that \( |\mathcal{P}_{2m-2}| = 4 < |\mathcal{P}_{2m-2}| = 5 \). Equal sharing, condition (a) for \( h = m - 2 \), and positive spillovers together imply that...
\[ u_j(P^N) = u_i(P^N) > u_i(P_{2m-2-1}) > u_j(P_{2m-2-2}) \text{ for all } i \in S \text{ such that } S \subseteq P_{2m-2-1} \text{ and } |S| = 2^{m-3} \text{ and all } j \in S \text{ such that } S \subseteq P_{2m-2-2} \text{ and } |S| = 2^{m-3}. \]

For \( t = 2m-2 \), the payoff of every member in two \( 2m-2 \)-person coalitions under \( P_{2m-2} \) is smaller than that under \( P^N \) by condition (a) for \( h = m - 1 \). Remark that \( |P_{2m-2}| = 3 \).

For \( t = 2m-2 + 1 \), the payoff of every member in the two coalitions under \( P_{2m-2+1} \) is smaller than that under \( P^N \). Remark that \( |P_{2m-2+1}| = 2 \). Equal sharing, condition (b), and negative association together imply that \( u_j(P^N) = u_i(P^N) > u_i(P_{2m-2+1}) \geq u_j(P_{2m-2+1}) \) for all \( i \in S \) such that \( |S| = 2^{m-1} \) and all \( i \in T \) such that \( |T| = 2^{m-1} + 1 \).

Therefore, the \( k^* \)-th stage coalition structure \( P_1 = P^* \) is sequentially dominated by \( P^N \).

**(Step 2)** Every \( k^* \)-th stage coalition structure is sequentially dominated by \( P^N \).

Take any \( k^* \)-th stage coalition structure \( P \). Here \( k^* = 2^{m-2} + 2 \).

First we consider a sequence \{\( P_t \}_{t=0}^T \) such that
1) \( P_0 = P = \{ r_1; r_2; r_3; \ldots; r_{k^*}; r_{k^*+1}; r_{k^*+2} \} \ (r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_{k^*}; r_{k^*+1} \leq r_{k^*+2}) \).
2) \( P_T = \{ 1; 1; 1; \ldots; 1; n - k^* + 1 \} \), where \( |P_T| = k^* \).
3) If \( t \) is zero or even, then a single player belonging to the largest coalition in \( P_t \) deviates and forms one person coalition in \( P_{t+1} \).
4) If \( t \) is odd, then the largest and the second largest coalitions in \( P_t \) merge into one coalition in \( P_{t+1} \).

Then the sequence \{\( P_t \}_{t=0}^T \) of coalition structures is given by:
\[
\begin{align*}
P_0 &= \{ r_1; r_2; r_3; \ldots; r_{k^*}; r_{k^*+1}; r_{k^*+2} \} \ (k^*-th \ stage) \\
\rightarrow P_1 &= \{ 1; r_1; r_2; r_3; \ldots; r_{k^*}; r_{k^*+1}; r_{k^*+2} \} \ (k^*+1-th \ stage) \\
\rightarrow P_2 &= \{ 1; r_1; r_2; r_3; \ldots; r_{k^*}; r_{k^*+1} + r_{k^*+2} - 1 \} \ (k^*-th \ stage) \\
\rightarrow P_3 &= \{ 1; r_1; r_2; r_3; \ldots; r_{k^*}; r_{k^*+1} + r_{k^*+2} - 2 \} \ (k^*+1-th \ stage) \\
\rightarrow \ldots \rightarrow \\
\rightarrow P_{T-2} &= \{ 1; 1; 1; \ldots; 1; r_1; \sum_{k=2}^{k^*} r_k - k^* + 2 \} \ (k^*-th \ stage) \\
\rightarrow P_{T-1} &= \{ 1; 1; 1; \ldots; 1; r_1; \sum_{k=2}^{k^*} r_k - k^* + 1 \} \ (k^*+1-th \ stage) \\
\rightarrow P_T &= \{ 1; 1; 1; \ldots; 1; \sum_{k=1}^{k^*} r_k - k^* + 1 \} \ (k^*-th \ stage)
\end{align*}
\]

Next consider \{\( P_t \}_{t=T+1}^{T'+T} \} such that
1) \( P_T = \{ 1; 1; 1; \ldots; 1; n - k^* + 1 \} \),
2) \( P_{T+T'} = P^* = \{ 1; 2; 2; 2; \ldots; 2; 2; 2m-1 + l \} \).
3) If \( t = T + \lambda \) and \( \lambda \) is 0 or even, then a single player in the largest coalition in \( P_{T+\lambda} \) deviates and form a singleton in \( P_{T+\lambda+1} \).
4) If \( t = T + \lambda \) and \( \lambda \) is odd, then two singletons in \( P_{T+\lambda} \) merge into one coalition in \( P_{T+\lambda+1} \).
This sequence \( \{\mathcal{P}_t\}_{t=0}^{T_T} \) of coalition structures is given by:
\[
\mathcal{P}_T = \{1; 1; 1; 1; \ldots; 1; 1; 1; n-k^* + 1\} \quad (k^*-th \text{ stage})
\]
- \( \mathcal{P}_{T+1} = \{1; 1; 1; 1; \ldots; 1; 1; 1; n-k^*\} \quad ((k^*+1)-th \text{ stage})
- \( \mathcal{P}_{T+2} = \{1; 1; 1; 1; \ldots; 1; 1; 2; n-k^*\} \quad (k^*-th \text{ stage})
- \( \mathcal{P}_{T+3} = \{1; 1; 1; 1; \ldots; 1; 1; 2; n-k^* - 1\} \quad ((k^*+1)-th \text{ stage})
- \( \mathcal{P}_{T+4} = \{1; 1; 1; 1; \ldots; 1; 2; 2; n-k^* - 1\} \quad k^*-th \text{ stage})
- \( \mathcal{P}_{T+5} = \{1; 1; 1; 1; \ldots; 1; 1; 2; 2; n-k^* - 2\} \quad ((k^*+1)-th \text{ stage})
- \( \ldots \rightarrow \ldots \)
- \( \mathcal{P}_{T+T'-2} = \{1; 1; 2; \ldots; 2; 2; n-2k^* + 3\} \quad k^*-th \text{ stage})
- \( \mathcal{P}_{T+T'-1} = \{1; 1; 1; 2; \ldots; 2; 2; n-2k^* + 4\} \quad (k^*+1)-th \text{ stage})
- \( \mathcal{P}_{T+T'} = \{1; 1; 2; 2; \ldots; 2; 2; n-2k^* + 4\} = \{1; 1; 2; 2; \ldots; 2; 2; 2m-1 + l\}
\]

By combining two sequences \( \{\mathcal{P}_t\}_{t=0}^{T_T} \) and \( \{\mathcal{P}_t\}_{t=T'}^{T'} \), we get a sequence \( \{\mathcal{P}_t\}_{t=0}^{T+T'} \) from any \( k^*-th \text{ stage} \) coalition structure \( \mathcal{P} = \mathcal{P}_0 \) to \( \mathcal{P}^* = \mathcal{P}_{T+T'} \).

For \( t = 0, 2, 4, \ldots, T, T+2, T+4, \ldots, T+T'-2 \), a single player belonging to the largest coalition in \( \mathcal{P}_t \) deviates and forms one person coalition \( \mathcal{P}_{t+1} \). The payoff of this single deviating player under \( \mathcal{P}^N \) is larger than that under \( \mathcal{P} \) because equal sharing, negative association, and efficiency together imply that for all \( \mathcal{P} \in \Pi \) with \( \mathcal{P} \neq \mathcal{P}^N \),

\[
u_i(\mathcal{P}^N) > \nu_i(\mathcal{P}) \quad \text{for all } i \in S \text{ such that } S \in \mathcal{P} \text{ and } |S| \geq |T| \quad \text{for all } T \in \mathcal{P} \quad (A.1)
\]

For \( t = 1, 3, 5, \ldots, T-1 \), the largest coalition \( S_1 \) and the second largest coalition \( S_2 \) in \( \mathcal{P}_t \) merge into one coalition in \( \mathcal{P}_{t+1} \). By condition (a) for \( h = 1 \), the payoff of every singleton player under any coalition structure \( \mathcal{P} \) with \( |\mathcal{P}| = 2m^2 + 2 = k^* \) is smaller than that under \( \mathcal{P}^N \). Since \( |\mathcal{P}_t| = k^* + 1 = 2m^2 + 2 > k^* \), it follows from positive spillovers that any singleton player prefers his/her payoff under \( \mathcal{P}^N \) to that under \( \mathcal{P}_t \), that is,

\[
u_i(\mathcal{P}^N) > \nu_i(\mathcal{P}_t) \quad \text{for all } i \in S \text{ such that } S \in \mathcal{P}_t \text{ and } |S| = 1. \quad (A.2)
\]

This fact together with equal sharing and negative association imply that \( \nu_j(\mathcal{P}^N) = \nu_k(\mathcal{P}^N) = \nu_i(\mathcal{P}^N) > \nu_i(\mathcal{P}_t) \geq \nu_j(\mathcal{P}_t) \geq \nu_k(\mathcal{P}_t) \) for all \( i \in S \) with \( |S| = 1 \), all \( j \in S_1 \), and all \( k \in S_2 \), where \( |S_1| \geq |S_2| \geq 1 \). In other words, the payoff of each player belonging to the merged coalition \( S_1 \cup S_2 \) under \( \mathcal{P}_t \) is smaller than that under \( \mathcal{P}^N \).

For \( t = T+1, T+3, T+5, \ldots, T+T'-1 \), two singleton coalitions in \( \mathcal{P}_t \) merge into one coalition in \( \mathcal{P}_{t+1} \). Since \( |\mathcal{P}_t| = k^* + 1 = 2m^2 + 3 > k^* \), we again have the above inequalities (A.2). Therefore, each member in the merged coalition receives a higher payoff under \( \mathcal{P}^N \) than under \( \mathcal{P}_t \).

Moreover, by Step 1, \( \mathcal{P}_{T+T'} = \mathcal{P}^* \) is sequentially dominated by \( \mathcal{P}^N \). These facts together imply that every \( k^*-th \text{ stage} \) coalition structure \( \mathcal{P} = \mathcal{P}_0 \) is sequentially dominated by \( \mathcal{P}^N \).

**Step 3** Every \( k \)-th coalition structure \( \mathcal{P} \) with \( 1 < k < k^* = 2m^2 + 2 \) is sequentially dominated by \( \mathcal{P}^N \).
Take any coalition structure $\mathcal{P}$ of less than $k^*$ coalitions other than $\mathcal{P}^N$. Consider the following sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $k^*$-th stage coalition structure $\mathcal{P}^*$. One person in a coalition of the maximal size $P_t$ deviates and forms a singleton in $\mathcal{P}_{t+1}$. By (A.1) in Step 2, each person in a coalition of the maximal size in $\mathcal{P}_t$ prefers his/her payoff under $\mathcal{P}^N$ to that under $\mathcal{P}_t$. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^*$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$, the sequence from $\mathcal{P}'$ to $\mathcal{P}^*$ in Step 2, and the sequence from $\mathcal{P}^*$ to $\mathcal{P}^N$ in Step 1. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 4) Every $k$-th coalition structure $\mathcal{P}$ with $n \geq k > k^* = 2^{m-2} + 2$ is sequentially dominated by $\mathcal{P}^N$.

Take any $k$-th stage coalition structure $\mathcal{P}$ such that $n \geq k > k^* = 2^{m-2} + 2$. Condition (a) for $h = 1$ together with positive spillovers imply that the payoff of every singleton player under $\mathcal{P}$ is smaller than that under $\mathcal{P}^N$. This fact and negative association together yield that the payoff of every player in any coalition under $\mathcal{P}$ is smaller than that under $\mathcal{P}^N$.

Consider any sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $k^*$-th stage coalition structure $\mathcal{P}'$ such that two coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$. Notice that each member in these two coalitions in $\mathcal{P}_t$ prefers his/her payoff under $\mathcal{P}^N$ to that under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$, the sequence from $\mathcal{P}'$ to $\mathcal{P}^*$ in Step 2, and the sequence from $\mathcal{P}^*$ to $\mathcal{P}^N$ in Step 1. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

Steps 1-4 show that every coalition structure other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$. Q.E.D.

7.2 Proof of Theorem 2

In what follows, we denote a coalition structure by $\mathcal{P} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$. We begin by proving the following lemma:

Lemma 1. Let $\mathcal{P} \in \Pi$ be given. Then we have: (1) $u_i(\mathcal{P}) < u_i^*(\mathcal{P}^N)$ for all $i \in N$ if and only if $B(k) < r_1/n$; (2) $u_i(\mathcal{P}) < u_i^*(\mathcal{P}^N)$ for all $i \in S_2 \cup S_3 \cup ... \cup S_k$ if and only if $B(k) < r_2/n$; (3) $u_i(\mathcal{P}) < u_i^*(\mathcal{P}^N)$ for all $i \in S_3 \cup S_4 \cup ... \cup S_k$ if and only if $B(k) < r_3/n$; ...; and (k) $u_i(\mathcal{P}) < u_i^*(\mathcal{P}^N)$ for all $i \in S_k$ if and only if $B(k) < r_k/n$.

Proof. It is easy to see that $u_i(\mathcal{P}) = m^*_S(\mathcal{P})/r_j = [f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})]/(r_jk^2)$ for $i \in S_j$ and $j = 1, ..., k$ (see Theorem 1 in Funaki and Yamato (1999)). Notice that for the grand coalition structure $\mathcal{P}^N$, $k = 1$ and $r_1 = n$, so that $u_i(\mathcal{P}^N) = [f(x^*_N(\mathcal{P}^N)) - f'(x^*_N(\mathcal{P}^N))x^*_N(\mathcal{P}^N)]/n$ for $i \in N$. We also remark that a player belonging to the smallest coalition, $S_1$, obtains the highest payoff among all players, that is, the payoff of each player $i$, $u_i(\mathcal{P})$, is less than or equal to $u_j(\mathcal{P}) = m^*_S(\mathcal{P})/r_1$ for $j \in S_1$. Therefore, $u_i(\mathcal{P}) < u_i^*(\mathcal{P}^N)$ for all $i \in N$ if and only if $B(k) = \{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})\}/[k^2\{f(x^*_N(\mathcal{P})^N) - f'(x^*_N(\mathcal{P}^N))x^*_N(\mathcal{P}^N)\}] < r_1/n$. 

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In similar, note that a player belonging to the smallest coalition, $S_2$, among the coalitions, $S_2, S_3, \ldots, S_k$, obtains the highest payoff among players in those $k-1$ coalitions, that is, the payoff of each player $i \in S_2 \cup S_3 \cup \ldots \cup S_k$, $u_i(P)$, is less than or equal to $u_j(P) = m_{S_j}(P)/r_j$ for $j \in S_2$. Hence, $u_i(P) < u_i(P^N)$ for all $i \in S_2 \cup S_3 \cup \ldots \cup S_k$ if and only if $B(k) = \{f(x_N^*(P)) - f'(x_N^*(P))x_N^*(P)\}/2^k \{f(x_N^*(P^N)) - f'(x_N^*(P^N))x_N^*(P^N)\} < r_2/n$. A similar argument works for the case of $B(k) < r_3/n, \ldots, B(k) < r_k/n$. Q.E.D.

It is easy to verify that any common pool resource game satisfies equal sharing, negative association, and positive spillovers. Next we prove that it also satisfies conditions (a) and (b).

Take any coalition structure $P$ with $|P| = 2$. Consider a case that $P = \{S_1, S_2\}$ such that $r_1 = 2m-1$ and $r_2 = 2m-1 + l$. Then $B(2) < 2m-1/n$ implies $B(2) < \frac{1}{n}$. Lemma 1 implies $u_i(P) < u_i(P^N)$ for all $i \in N$. This shows condition (b) in Theorem 1.

For any $h = 1, 2, \ldots, m - 1$, take any coalition structure $P$ with $|P| = 2m-h-1 + 2$. Suppose there is a coalition $S \in P$ such that $|S| = 2h-1$. Then $S$ is equal to some $S_p$ for $p, 1 \leq p \leq k$. It is not hard to see that for two coalition structures $P_k = \{S_1, S_2, \ldots, S_k\}$ and $P_{k'} = \{S'_1, S'_2, \ldots, S'_{k'}\}$ with $k < k'$, if $S \in P_k$ and $S \in P_{k'}$, then $m^*_S(P_k) > m^*_S(P_{k'})$ (see Theorem 2 in Funaki and Yamato (1999)). Since $B(k) = m^*_N(P)/m^*_N(P^N) = m^*_N(P)/[km^*_N(P^N)]$, it follows that $B(k)$ is monotonically decreasing in $k$. By this fact and the inequality $B(2m-h+2) < \frac{2h-1}{n}$, we have $B(2m-h+1) < \ldots < B(2m-h+3) < B(2m-h+2) < \frac{2h}{n}$, which implies $B(|P|) < \frac{2}{n}$. Lemma 1 implies $u_i(P) < u_i(P^N)$ for all $i \in S_p \cup S_{p+1} \cup \ldots \cup S_k$. This means that $u_i(P) < u_i(P^N)$ for all $i \in S$. Thus condition (a) in Theorem 1 is satisfied. Q.E.D.

### 7.3 Proof of Theorem 3

First of all, note that $B(k) = \frac{1}{2^k} \left(\frac{a-1+\alpha k}{\alpha k}\right)^{\alpha/(1-\alpha)}$. In addition, $\lim_{a \to 0} B(k) = 1/k^2$ for any $k$. Hence for sufficiently small $\alpha > 0$, $B(k)$ is very close to $1/k^2$.

Suppose that $n \geq 4$. Let $n = 2m + l$. Given $m \geq 2$, consider any integer $n \in [2^m, 2^{m+1})$. First we will show that $\lim_{a \to 0} B(2m-h+1) = 1/(2^m-h+1+2)^2 < 2h/n$ for $h = 1, \ldots, m - 1$. Since $2^h/n > 2^h/2^{m+1} = 1/2^{m-h+2}$, it is sufficient to prove that $1/(2^m-h+2)^2 < 1/2^{m-h+2}$, that is, $(2^m-h+1)^2 > 2^{m-h+2}$. If $h \leq m - 4$, then $2(2^m-h+1) \geq 2^m-h+2$, implying the desired result. Also, for $h = m - 3$, $(2^m-h+2)^2 = (2^2 + 2)^2 > 2^2 = 2^{m-h+2}$,

for $h = m - 2$, $(2^m-h+1)^2 = (2+2)^2 = 4^2 = 2^{m-h+2}$, and for $h = m - 1$, $(2^m-h+2)^2 = (1+2)^2 = 3^2 > 2^3 = 2^{m-h+2}$.

Moreover, $\lim_{a \to 0} B(2) = 1/4 = 2^{m-1}/2^{m+1} < 2^{m-1}/n$. Also, it is clear that $B(k)$ is increasing in $k$. Moreover, $B(k)$ is an increasing function of $\alpha$. It follows from Theorem 2 that there exists $\alpha^* \in (0, 1)$ such that the grand coalition structure $P^N$ is sequentially stable for $\alpha < \alpha^*$. For $n \leq 3$, see Table 1 in Section 4.2 indicating the region of $\alpha$ for which $P^N$ is sequentially stable. Q.E.D.
7.4 Proof of Proposition 2

We will show that any coalition structure containing three coalitions is not sequentially dominated by \( P^{N\setminus \{i\}} \) if \( C(3) \geq \frac{9}{8} \). Let \( P = \{S_1, S_2, S_3\}, |S_1| \leq |S_2| \leq |S_3| \), be a coalition structure containing 3 coalitions. In any sequence from \( P \) to \( P^{N\setminus \{i\}} \), two coalitions must merge into one coalition. Thus it is enough to show that the payoff of each player in one of two coalitions is smaller than the payoff in the coalition \( N \setminus \{i\} \) of \( P^{N\setminus \{i\}} \). Hence, if the largest payoff of a player in the second largest \( S_2 \) among all coalition structures with 3 coalitions is smaller than the payoff of a player in \( N \setminus \{i\} \), we can attain our purpose. Then we have to compare the payoff \( m^*_j(P) \) of player \( j \) in a coalition \( S_2 \) of the smallest size with the payoff \( m^*_j(P^{N\setminus \{i\}}) \).

Remark that such a coalition structure is given by \( |S_1| = 1 \) and \( |S_2| = |S_3| = (n - 1)/2 \) if \( n \) is odd, and \( |S_1| = 1, |S_2| = (n - 2)/2, |S_3| = (n + 2)/2 \) if \( n \) is even. The payoff of each player \( j \in S_2 \) is given by \( m^*_j(P) = [f(x^*_N(P)) - f'(x^*_N(P))x^*_N(P)]/(9r_2) \), and the payoff of each player \( j \in N \setminus \{i\} \) is given by \( m^*_j(P^{N\setminus \{i\}}) = [f(x^*_N(P^{N\setminus \{i\}})) - f'(x^*_N(P^{N\setminus \{i\}}))x^*_N(P^{N\setminus \{i\}})]/(4(n - 1)) \).

Note that for \( j \in S_2 \), \( m^*_j(P) \geq m^*_j(P^{N\setminus \{i\}}) \) iff \( [4(n - 1)/(9r_2)]f(x^*_N(P)) - f'(x^*_N(P))x^*_N(P)}/[f(x^*_N(P^{N\setminus \{i\}})) - f'(x^*_N(P^{N\setminus \{i\}}))x^*_N(P^{N\setminus \{i\}})] \geq [4(n - 1)/(9r_2)]C(3) \geq 1 \).

There are two cases to examine. First, if \( n \) is even, then for \( P \in \Pi \) with \( r_2 = (n - 2)/2, 4(n - 1)/(9r_2) = 8(n - 1)/9(n - 2) \), so that if \( C(3) \geq 9/8 \), then \( m^*_j(P) > m^*_j(P^{N\setminus \{i\}}) \). Second, if \( n \) is odd, then for \( P \in \Pi \) with \( r_2 = (n - 1)/2, 4(n - 1)/(9r_2) = 8/9 \), so that if \( C(3) \geq 9/8 \), then \( m^*_j(P) \geq m^*_j(P^{N\setminus \{i\}}) \). Q.E.D.

7.5 Proof of Corollary 2

It is easy to see that \( C(3) = \left( \frac{3(\alpha + 1)}{2(\alpha + 2)} \right) ^{-\alpha/(1-\alpha)} \). It is not hard to check that \( 1/C(3) = \left( \frac{3(\alpha + 1)}{2(\alpha + 2)} \right) ^{\alpha/(1-\alpha)} < \frac{8}{5} \) if \( \alpha \geq 0.583804 \). Q.E.D.

7.6 Proof of Theorem 4

The proof consists of three steps.

Step 1: Take any coalition structure \( P \) with \( |P| = m \). Every player gets a higher payoff under the grand coalition structure \( P^N \) than under \( P \) because for all \( i \in N \), \( u_i(P^N) = \frac{1-\alpha}{n} \left( \frac{\alpha}{q} \right) ^{\alpha/(1-\alpha)} \geq \frac{1-\alpha}{m} \left( \frac{\alpha-1-m}{mq} \right) ^{\alpha/(1-\alpha)} = \max_{j \in N} u_j(P') \geq \max_{j \in N} u_j(P) \geq u_i(P) \), where \( P' \) is a coalition structure containing at least one singleton coalition of only one player and \( |P'| = m.9 \) Therefore, \( P^N \) sequentially dominates \( P \) when mergers of \( m \) coalitions are possible.

Step 2: Take any coalition structure \( P \) with \( m + 1 \leq |P| \leq n \). Notice that for every coalition structure \( P \) with \( |P| = k \), there are \( S', S'' \in P \) and \( S \in P' \) such that

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9For example, consider \( P' = \{1;1;1, n-m+1\} \), that is, the coalition structure consisting of \( m - 1 \) one-person coalitions and one coalition of \( n - m + 1 \) persons. Note that each player in the one-person coalitions get the highest payoff, \( \frac{1-\alpha}{m} \left( \frac{\alpha-1-m}{mq} \right) ^{\alpha/(1-\alpha)} \), under \( P' \).
$S' \cup S'' = S$ and $|P'| = k - 1$. Therefore, there are a coalition structure $P'$ with $|P'| = \overline{m}$ and a sequence of coalition structures from $P$ to $P'$ through bilateral mergers of two coalitions at each step. Moreover, for all $S, S' \in P$ and all $i \in S \cup S'$, $u_i(P) < u_i(P')$, because $u_i(P_N) = \frac{1-\alpha}{n} \left( \frac{a}{q} \right)^{\alpha/(1-\alpha)} > \frac{1-\alpha}{m^2} \left( \frac{a-1+m}{m} \right)^{\alpha/(1-\alpha)} > \frac{1-\alpha}{|P|^2} \left( \frac{a-1+|P|}{|P|^q} \right)^{\alpha/(1-\alpha)}$

max$_{j \in N} u_j(P'') \geq$ max$_{j \in N} u_j(P) \geq u_i(P)$, where $P''$ is a coalition structure containing at least one coalition of only one player and $|P''| = |P| > \overline{m}$. That is, the payoff of each member in any pair of coalitions under $P$ is smaller than that under $P_N$. By combining the above sequence and the sequence in Step 1, we have the sequence from $P$ to $P_N$ and $P_N$ sequentially dominates $P$ when mergers of two coalitions are possible.

Step 3: Take any coalition structure $P$ with $2 \leq |P| \leq \overline{m}-1$. Consider a sequence of coalition structures from $P$ to a coalition structure $P'$ with $|P'| = \overline{m}$, $\{P_t\}_{t=0}^{m^*-2}$, such that a single player belonging to the largest coalition in $P_t$ deviates and forms one person coalition in $P_{t+1}$ at each step. By using Lemma 1 in the appendix, it is easy to check that this person in a coalition of the maximize size in $P_t$ prefers his/her payoff under $P_N$ to that under $P_t$. By combining the above sequence and the sequence in Step 1, we have the sequence from $P$ to $P_N$ and $P_N$ sequentially dominates $P$ when mergers of $m^*$ coalitions and breakups into two coalitions are possible.

Therefore, $P_N$ is $\overline{m} - \overline{b}$ sequentially stable. Q.E.D.

7.7 Proof of Theorem 5

Step 1: Take any coalition structure $P$ with $|P| = \overline{m}$. Every firm gets a higher payoff under $P_N$ than under $P$ because for all $i \in N, u_i(P_N) = \frac{1}{|P|^2} > \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{|P|^2} \left( \frac{|P|}{|P|+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{|P|m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{|P|m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{|P|m}{m+1} \right)^2 = $ max$_{j \in N} u_j(P') \geq$ max$_{j \in N} u_j(P) \geq u_i(P)$, where $P'$ is a coalition structure containing at least one singleton coalition of only one player and $|P'| = \overline{m}$. Therefore, $P_N$ sequentially dominates $P$ when mergers of $m$ coalitions are possible.

Step 2: Take any coalition structure $P$ with $\overline{m} + 1 \leq |P| \leq n$. It is easy to construct a sequence of coalition structures from $P$ to some coalition structure $P'$ with $|P'| = \overline{m}$ through bilateral mergers of two coalitions at each step. Moreover, for all $S, S' \in P$ and all $i \in S \cup S'$, $u_i(P) < u_i(P')$, because $u_i(P_N) = \frac{1}{|P|^2} > \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = \frac{1}{\overline{m}} \left( \frac{m}{m+1} \right)^2 = $ max$_{j \in N} u_j(P'') \geq$ max$_{j \in N} u_j(P) \geq u_i(P)$, where $P''$ is a coalition structure containing at least one coalition of only one player and $|P''| = |P| > \overline{m}$.

By combining the above sequence and the sequence in Step 1, we have the sequence from $P$ to $P_N$ and $P_N$ sequentially dominates $P$ when mergers of two coalitions are possible.

We omit the rest of the proof that is similar to Step 3 in the proof of Theorem 4. Q.E.D.

References


Figure 1. Example 1.
Figure 2. Proof of Theorem 1: the Eight-Person Case