A Note on Lower Bound for Multiplicative Odds Theorem of Optimal Stopping

Tomomi Matsui*      Katsunori Ano†

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Department of Social Engineering,
Graduate School of Decision Science and Technology,
Tokyo Institute of Technology

*Department of Social Engineering, Graduate School of Decision Science and Technology, Tokyo Institute of Technology 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8550, Japan.
†Department of Mathematical Sciences, Shibaura Institute of Technology, Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570 Japan.
A Note on Lower Bound for Multiplicative Odds Theorem of Optimal Stopping

Katsunori Ano* Tomomi Matsui†

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Abstract

This note provides a bound of the optimal maximum probability for the multiplicative odds theorem in optimal stopping theory. We deal with an optimal stopping problem which maximizes the probability of stopping on any of the last \( m \) successes of a sequence of independent Bernoulli trials of length \( N \), where \( m \) and \( N \) are predetermined integers satisfying \( 1 \leq m < N \). This problem is an extension of Bruss’ odds problem [1]. Tamaki [4] gave an optimal stopping rule. A lower bound of the optimal probability is shown in this note. It is interesting to see that our lower bound is attained by a variation of well-known secretary problem, which is a special case of odds problem.

keywords: optimal stopping; odd problem; lower bound; secretary problem; Maclaurin’s inequality

1 Introduction

Let \( X_1, X_2, \ldots, X_N \) be a sequence of independent Bernoulli random variables. The outcome of each random variable is either a success or a failure. We let

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*Department of Mathematical Sciences, Shibaura Institute of Technology, Fulasaku, Minuma-ku, Saitama-shi, Saitama 337-8570 Japan.
†Department of Information and System Engineering, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan.
$X_j = 1$ if $X_j$ is a success, and $X_j = 0$ otherwise. These random variables may be regarded as *indices* for an observation of underlying discrete stochastic process. For example, we can think them the record process. A decision maker observes sequentially $X_1, X_2, \ldots, X_N$ with the objective to predict, with the maximum probability, correctly the occurrence of any of the last $m$ successes at its respective occurrence time. We call the above problem a *multiplicative odds problem* of order $m$. We discuss asymptotic lower bounds of probability of “win” (i.e., obtaining any of the last $m$ successes).

When $m = 1$, the problem is equivalent to the well-known Bruss’ odds problem [1], which has an elegant and simple optimal stopping strategy known as *Odds theorem* or *Sum the Odds theorem*. A typical lower bound for an asymptotic optimal value (probability of win), when $N$ goes to infinity, is shown to be $e^{-1}$ in Bruss [2], which is equal to that for the classical secretary problem. One of the reason why odds problem is attractive in optimal stopping theory is to include the secretary problem as a special case.

For general case ($m \geq 2$), Tamaki [4] showed *Sum the Multiplicative Odds Theorem*, which gives an optimal stopping rule obtained by a threshold strategy. Tamaki [4] also discussed the celebrated secretary problem and derived an asymptotic optimal value.

In this note, we give an asymptotic lower bound of probability of win for the multiplicative odds problem. Our lower bound is equivalent to the asymptotic optimal value for secretary problem obtained by Tamaki in [4], which implies the tightness of our bound. A special feature of our proof is an application of Maclaurin’s inequality [3] to get our bound.

2 Preliminaries

For any pair of positive integers $k, N$ satisfying $1 \leq k \leq N$ and a vector $r \in \mathbb{R}^N$, $e_k(r)$ denotes a $k$-th elementary symmetric function of $r = (r_1, r_2, \ldots, r_N)$ defined by

$$e_k(r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} r_{i_1} r_{i_2} \cdots r_{i_k} = \sum_{B \subseteq \{1, 2, \ldots, N\}} \prod_{i \in B} r_i,$$

which is a sum of $\binom{N}{k}$ terms. We also define that $e_0(r) = 1$. The $k$-th elementary symmetric mean of $r$ is defined by

$$S_k(r) = \frac{e_k(r)}{\binom{N}{k}}.$$
Here we describe Maclaurin’s inequalities, which plays an important role in the next section.

**Lemma 1** (Maclaurin’s inequalities)\cite{3} Every non-negative vector $r \in \mathbb{R}^N_+$ satisfies the following chain of inequalities

$$S_1(r) \geq \sqrt{S_2(r)} \geq \sqrt[3]{S_3(r)} \geq \cdots \geq \sqrt[N]{S_N(r)}.$$

### 3 Lower Bound

We deal with a sequence of independent 0/1 random variables, $X_1, X_2, \ldots, X_N$, where $N$ is a given positive integer and the distribution is $\Pr[X_k = 1] = p_k$, $\Pr[X_k = 0] = 1 - p_k = q_k$, $0 \leq p_k < 1$ for each $k$. We define $r_k = p_k/q_k$ for each $k$. The $r_k$’s are called odds. A multiplicative odds problem of order $m$ finds a strategy to predict, with the maximum probability, correctly the occurrence of any of the last $m$ successes at its respective occurrence time.

First, we briefly review Sum the Multiplicative Odds Theorem shown by Tamaki in \cite{4}. An optimal stopping rule for multiplicative odds problem is obtained by a threshold strategy, that is, it stops on the first success for which the sum of the $m$-fold multiplicative odds of success for the future trials is less than or equal to 1. More precisely, the optimal rule stops on the first success $X_i = 1$ with

$$i \geq i_* \overset{\text{def}}{=} \min\{k \geq 1 \mid e_m(r_{k+1}, r_{k+2}, \ldots, r_N) \leq 1\}.$$  

The corresponding probability of win is equal to

$$q_i q_{i+1} \cdots q_N (e_m(\tilde{r}) + e_{m-1}(\tilde{r}) + \cdots + e_1(\tilde{r}))$$

where $\tilde{r} = (r_i, r_{i+1}, \ldots, r_N)$.

In the rest of this section, we discuss the probability of win for multiplicative odds problem under the above optimal stopping rule.

**Theorem 1** Let us consider a multiplicative odds problem of order $m$ defined on $X_1, X_2, \ldots, X_N$ satisfying $m \leq N$ and $e_m(r_1, r_2, \ldots, r_N) \geq 1$. Under the optimal stopping rule, the probability of win is greater that or equal to

$$\exp\left(-m!^{1/m}\sum_{k=1}^{m} \frac{(m!)^{k/m}}{k!}\right).$$
The objective function of $P_1$ becomes

$$e_m(r_2, r_3, \ldots, r_N) \leq 1 \leq e_m(r_1, r_2, r_3, \ldots, r_N).$$

Under assumption (1), the probability of win, denoted by $V_{m,N}$, is equal to $V_{m,N} = q_1 q_2 \cdots q_N(e_m(r) + e_{m-1}(r) + \cdots + e_1(r))$. Thus, the greatest lower bound of the probability of win is equal to the optimal value of the following optimization problem;

$$P_1: \quad \begin{array}{ll}
& \text{min.} \\
& V_{m,N} = q_1 q_2 \cdots q_N(e_m(r) + e_{m-1}(r) + \cdots + e_1(r)) \\
& \text{s. t.} \\
& 0 \leq r_k \quad (\forall k \in \{1, 2, \ldots, N\}), \\
& q_k = \frac{1}{1 + r_k} \quad (\forall k \in \{1, 2, \ldots, N\}), \\
& e_m(r_1, r_2, r_3, \ldots, r_N) \geq 1, \\
& e_m(r_2, r_3, \ldots, r_N) \leq 1,
\end{array}$$

composed of $N$ independent variables $(r_1, r_2, \ldots, r_N)$ and $N$ dependent variables $(q_1, q_2, \ldots, q_N)$ defined by $q_k = 1/(1 + r_k)$ for each $k \in \{1, 2, \ldots, N\}$.

In the rest of this paper, we denote $(r_2, r_3, \ldots, r_N)$ by $r_-$ for simplicity. The objective function of $P_1$ becomes

$$V_{m,N} = q_2 \cdots q_N \left( \sum_{j=1}^{m} q_1 e_j(r) \right) = q_2 \cdots q_N \left( \sum_{j=1}^{m} q_1 (e_j(r_-) + r_1 e_{j-1}(r_-)) \right)$$

$$= q_2 \cdots q_N \left( \sum_{j=1}^{m} q_1 e_j(r_-) + (1 - q_1) e_{j-1}(r_-) \right)$$

$$= q_2 \cdots q_N \left( q_1 e_m(r_-) - q_1 + \sum_{j=0}^{m-1} e_j(r_-) \right)$$

$$= q_2 \cdots q_N \left( e_m(r_-) - 1 \frac{e_m(r_-) - 1}{1 + r_1} + \sum_{j=0}^{m-1} e_j(r_-) \right).$$

If we fix variables $\{r_2, r_3, \ldots, r_N\}$ to values of a feasible solution of $P_1$, the minimum of $V_{m,N}$ is attained by setting the remained variable $r_1$ to a value defined by $\min\{r_1 \geq 0 \mid e_m(r) \geq 1\}$, since every feasible solution $r$ satisfies $e_m(r_-) - 1 \leq 0$. Thus, problem $P_1$ has an optimal solution $r^*$ satisfying that $e_m(r^*) = 1$. Maclaurin’s inequalities in Lemma 1 imply following
inequalities;

\[
\frac{e_1(r^*)}{(N_1)} \geq \sqrt{\frac{e_2(r^*)}{(N_2)}} \geq \sqrt[3]{\frac{e_3(r^*)}{(N_3)}} \geq \cdots \geq \sqrt[m]{\frac{e_m(r^*)}{(N_m)}} = \frac{1}{(N_m)^{1/m}}.
\]

Thus, we have that

\[
1 \leq \forall k \leq m, \quad e_k(r^*) \geq \left(\frac{N}{k}\right) - \frac{N}{m}.
\]

Similarly, Maclaurin’s inequalities imply that

\[
\frac{1}{(N_m)^{1/m}} = \sqrt[m]{\frac{e_m(r^*)}{(N_m)}}, \quad \sqrt[m+1]{\frac{e_{m+1}(r^*)}{(N_{m+1})}}, \ldots, \sqrt[N]{\frac{e_N(r^*)}{(N_N)}}
\]

and thus \( r^* \) satisfies

\[
m \leq \forall k \leq N, \quad \quad \frac{N}{k} - \frac{N}{m} \geq e_k(r^*).
\]

Let \( V_{m,N}^* \) be the optimal value of P1. From the above, we obtain an upper bound of \( 1/V_{m,N}^* \) as follows;

\[
\frac{1}{V_{m,N}^*} = \frac{(1 + r_1^*)(1 + r_2^*) \cdots (1 + r_N^*)}{\sum_{k=1}^{m} e_k(r^*)}
\]

\[
= \frac{1 + e_1(r^*) + e_2(r^*) + \cdots + e_m(r^*) + \cdots + e_N(r^*)}{e_1(r^*) + \cdots + e_m(r^*)}
\]

\[
= 1 + \frac{1 + e_{m+1}(r^*) + \cdots + e_N(r^*)}{e_1(r^*) + \cdots + e_m(r^*)}
\]

\[
\leq 1 + \frac{1}{\frac{N}{1} - \frac{N}{m}} + \frac{1}{\frac{N}{2} - \frac{N}{m}} + \cdots + \frac{1}{\frac{N}{m} - \frac{N}{m}}
\]

5
\[
1 + \sum_{k=1}^{N} \frac{\binom{N}{k}}{\binom{m}{N}} = \sum_{k=0}^{N} \frac{\binom{N}{k}}{\binom{m}{N}} = \left(1 + \left(\frac{N}{m}\right)^{\frac{1}{m}}\right)^N
\]
\[
= \sum_{k=1}^{m} \frac{\binom{N}{k}}{\binom{m}{N}} = \sum_{k=1}^{m} \frac{\binom{N}{k}}{\binom{m}{N}}.
\]

From the above, we obtain a lower bound of \(V_{m,N}^*\) denoted by
\[
V_{m,N}^* \geq \left(1 + \left(\frac{N}{m}\right)^{\frac{1}{m}}\right)^N.
\]

It is easy to see that problem P1 has an optimal solution \(r_1 = r_2 = \cdots = r_N = \left(\frac{N}{m}\right)^{\frac{1}{m}}\) whose corresponding objective value attain the above lower bound, and thus
\[
V_{m,N}^* = \sum_{k=1}^{m} \frac{\binom{N}{k}}{\binom{m}{N}} \geq \exp\left(-N\left(\frac{N}{m}\right)^{\frac{1}{m}}\right) \sum_{k=1}^{m} \frac{\binom{N}{k}}{\binom{m}{N}}.
\]

Lastly, we consider a lower bound independent of \(N\). It is easy to see that the optimal value \(V_{m,N}^*\) of P1 is non-increasing with respect to \(N\). Thus, \(\lim_{N \to \infty} V_{m,N}^*\) gives a general lower bound. Since
\[
-N\left(\frac{N}{m}\right)^{-\frac{1}{m}} = -N \left(\frac{N!}{(N-m)!m!}\right)^{-\frac{1}{m}} = -(m!)^{-\frac{1}{m}} \left(\frac{N!(N-m)!}{N!}\right)^{\frac{1}{m}}
\]
\[
= -(m!)^{-\frac{1}{m}} \left(\frac{1}{1 - \frac{0}{N}} \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{m-1}{N}\right)\right)^{\frac{1}{m}}
\]
\[
\to -(m!)^{-\frac{1}{m}}, \text{ as } N \to \infty,
\]
and
\[
\frac{\binom{N}{k}}{\binom{N}{m}} \to \frac{(m!)^{\frac{1}{m}}}{k!} \cdot \frac{N!}{N^k(N-k)!} \cdot \left(\frac{(N-m)!N^m}{N!}\right)^{\frac{1}{m}}
\]
\[
\begin{align*}
&= \frac{(m!)^k}{k!} \cdot \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{k-1}{N}\right) \cdot \left(\frac{1}{(1 - \frac{1}{N}) \cdots (1 - \frac{m-1}{N})}\right)^{\frac{1}{m}} \\
&\rightarrow \frac{(m!)^k}{k!}, \text{ as } N \to \infty,
\end{align*}
\]

we reach that

\[
\lim_{N \to \infty} V_{m,N}^* \geq \exp(-\frac{(m!)^{1/m}}{m!}) \sum_{k=1}^{m} \frac{(m!)^{k/m}}{k!}.
\]

The above theorem tells us very interesting result that our lower bound of probability of win for multiplicative odds problem is attained by the one for the corresponding secretary problem (shown by Tamaki in [4]), which is a special case of multiplicative odds problem.

\section*{References}


