Compare the Ratio of Symmetric Polynomials of Odds to One and Stop

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Discussion Paper No 2014-05
September 11, 2014

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Abstract

In this paper, we deals with an optimal stopping problem that maximizes the probability of selecting \( k \) out of the last \( \ell \) success, given a sequence of independent Bernoulli trials of length \( N \), where \( k \) and \( \ell \) are predetermined integers satisfying \( 1 \leq k \leq \ell < N \). This problem includes some natural problems as special cases, e.g., Bruss’ odds problem, Bruss and Paindaveine’s problem of selecting the last \( \ell \) successes, and Tamaki’s multiplicative odds problem for stopping at any of the last \( m \) successes. We show that an optimal stopping rule is obtained by a threshold strategy. We also present the tight lower bound and an asymptotic lower bound of the probability of win. Interestingly, our asymptotic lower bound is attained using a variation of the well-known secretary problem, which is a special case of the odds problem. Our approach is based on application of Newton’s inequalities and optimization technique, which gives a unified view to the previous works.

keywords: optimal stopping; odd problem; lower bound; secretary problem; Newton’s inequality

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1 Introduction

Let $X_1, X_2, \ldots, X_N$ denote a sequence of independent Bernoulli random variables. The outcome of each random variable is either a *success* or a *failure*. We let $X_j = 1$ if $X_j$ is a success, and $X_j = 0$ otherwise. These random variables can be regarded as *indices* for the observation of an underlying discrete stochastic process. For example, we can assume them to constitute the record process. This paper deals with an optimal stopping problem of maximizing the probability of selecting $k$ out of the last $\ell$ successes where $1 \leq k \leq \ell < N$. More precisely, the problem may be stated as follows.

We consider a game that a player is given the digits (realization of random variables) one by one and allowed to select the variable when he observes a success. The number of selected variables must be less than or equal to $k$. The player wins if he selected exactly $k$ variables contained in the set of last $\ell$ successes. For example, consider a case with $N = 8$, $k = 3$ and $\ell = 4$. When $(X_1, X_2, \ldots, X_8)$ has a vector of realized values $(0, 1, 1, 0, 0, 1, 1, 1)$, the player wins if he selected exactly 3 variables in the set \{X_3, X_6, X_7, X_8\}. We deals with a problem of maximizing a probability of win. It is easy to see that the player wins if and only if the first selected variable is in \{X_3, X_6\} by simply enumerating following $k = 3$ successes. Under this strategy, the player wins if the set of selected variables is either \{X_3, X_6, X_7\} or \{X_6, X_7, X_8\}. Thus, the player only need to observe the sequence with an objective to correctly predict the occurrence of the $m$-th last success satisfying $k \leq m \leq \ell$. From the above, the problem becomes a single stopping problem of maximizing the probability of stopping on a random variable $X_m$ satisfying $X_m = 1$ and $\ell \geq X_m + X_{m+1} + \cdots + X_N \geq k$. We present an optimal strategy and an asymptotic lower bound of the probability of "win" (i.e., obtaining $m$-th last success with $k \leq m \leq \ell$).

Although our problem setting has artificial fashion, it includes some natural problems as special cases (see Table 1). When $\ell = k = 1$, the problem is equivalent to the well-known Bruss’ odds problem [1], which has an elegant and simple optimal stopping strategy known as the *Odds theorem* or *Sum-the-Odds theorem*. A typical lower bound for an asymptotic optimal value (the probability of win), when $N$ approaches infinity, has been shown to be $e^{-1}$ by Bruss [2], which is equal to that for the classical secretary problem. One of the reason why the odds problem is popular in the optimal stopping theory is that it includes the secretary problem as a special case. If $\ell = k \geq 1$, Bruss and Paindaveine [3] showed that an optimal stopping rule is
Table 1: Previous results and our results.

<table>
<thead>
<tr>
<th>model</th>
<th>condition</th>
<th>lower bound</th>
<th>key inequality (⋆)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bruss [1]</td>
<td>ℓ = k = 1</td>
<td>$e^{-1}$</td>
<td>$\frac{r_i + r_{i+1} + \cdots + r_N}{e_{\ell}(r)} &lt; 1$ [2]</td>
</tr>
<tr>
<td>B&amp;P(†) [3]</td>
<td>ℓ = k ≥ 1</td>
<td>$\frac{\ell^\ell}{(\ell)!} e^\ell$ (⋆)</td>
<td>$\frac{e_\ell(r)}{e_{\ell-1}(r)} &lt; 1$ [3]</td>
</tr>
<tr>
<td>Tamaki [7]</td>
<td>ℓ ≥ k = 1</td>
<td>$\exp\left(-\frac{1}{\ell!}\right) \sum_{m=1}^{\ell} \frac{\ell! r^m}{m!}$ (⋆)</td>
<td>$\frac{e_\ell(r)}{e_0(r)} &lt; 1$ [5]</td>
</tr>
<tr>
<td>this paper</td>
<td>ℓ ≥ k ≥ 1</td>
<td>(‡) see below (⋆)</td>
<td>$\frac{e_\ell(r)}{e_{k-1}(r)} &lt; 1$ (⋆)</td>
</tr>
</tbody>
</table>

† Bruss and Paindaveine.
‡ $\exp\left(-\frac{1}{(k-1)!}\right) \sum_{m=k}^{\ell} \frac{1}{m!} \left(\frac{\ell}{(k-1)!}\right)^{\frac{m}{k-1+m}}$.

* Results obtained in this paper.

* An optimal strategy is attained by a threshold strategy defined by the minimum index $i$ satisfying the key inequality in the last column (see (2) for detail) where $r = (r_i, r_{i+1}, \ldots, r_N)$ and other notations are defined by (1).

obtained by a threshold strategy. When $\ell ≥ k = 1$, Tamaki [7] demonstrated the Sum-the-Multiplicative-Odds theorem, which gives an optimal stopping rule obtained using a threshold strategy. Recently, we discussed his model and showed a lower bound of the probability of win [5]. Bruss and Paindaveine [3] and Tamaki [7] also discussed the secretary problem and derived asymptotic optimal values.

In this paper, we describe an optimal strategy and derive the greatest lower bound of the probability of win for the the problem of selecting $k$ out of the last $\ell$ success. The asymptotic value of our lower bound is equivalent to the asymptotic optimal value for the secretary problem obtained by Bruss [2], Bruss and Paindaveine [3], and Tamaki [7]. A special feature of our proof is application of Newton’s inequalities [6] and optimization technique to obtain our bound.
2 Elementary Symmetric Polynomials

For any pair of positive integers $m, N$ satisfying $1 \leq m \leq N$ and a vector $\mathbf{r} \in \mathbb{R}^N$, $e_m(\mathbf{r})$ denotes the $m$-th elementary symmetric polynomial (function) of $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ defined by

$$e_m(\mathbf{r}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} r_{i_1}r_{i_2}\cdots r_{i_m} = \sum_{B \subseteq \{1, 2, \ldots, N\} \text{ and } |B| = m} \prod_{i \in B} r_i,$$

which is the sum of $\binom{N}{m}$ terms. We also define that $e_0(\mathbf{r}) = 1$. The $m$-th elementary symmetric mean of $\mathbf{r}$ is defined by

$$S_m(\mathbf{r}) = \frac{e_m(\mathbf{r})}{\binom{N}{m}} \quad (\forall m \in \{1, 2, \ldots, N\}) \text{ and } S_0(\mathbf{r}) = 1.$$

We abbreviate $S_m(\mathbf{r})$ to $S_m$, when there is no ambiguity. The elementary symmetric polynomials satisfy the following inequalities shown by Newton.

**Theorem 1** (Newton’s inequalities [6]) For every non-negative vector $\mathbf{r} \in \mathbb{R}^N_+$ and a positive integer $1 < m < N$,

$$S_m(\mathbf{r})^2 \geq S_{m-1}(\mathbf{r})S_{m+1}(\mathbf{r}),$$

with equality exactly when all the $r_i$ are equal.

Newton’s inequalities directly implies the followings.

**Lemma 1** For any positive vector $\mathbf{\tilde{r}} = (\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N) > 0$ and integers $(m, \ell)$ satisfying $1 \leq m \leq \ell \leq N$, the inequality $\frac{e_{\ell}(\mathbf{\tilde{r}})}{e_{m-1}(\mathbf{\tilde{r}})} \geq \frac{e_{\ell}(\mathbf{\tilde{r}})}{e_{m}(\mathbf{\tilde{r}})}$ holds.

**Proof.** The positivity of $\mathbf{\tilde{r}}$ implies that $S_m(\mathbf{\tilde{r}}) > 0$ ($0 \leq \forall m' \leq N$). Newton’s inequalities is equivalent to the midpoint log-concavity $\log(S_m) \geq (1/2)(\log(S_{m-1}) + \log(S_{m+1}))$, which directly yields the concavity of a se-
sequence \((\log(S_0), \log(S_1), \log(S_2), \ldots, \log(S_N))\) and the following inequalities:

\[
\frac{\log(S_m) + \log(S_{t-1})}{2} \geq \frac{\log(S_{m-1}) + \log(S_t)}{2},
\]

\[
\frac{S_{m}S_{t-1}}{S_{m-1}S_{t}} \geq S_{m-1}S_{t},
\]

\[
\left(\frac{N}{m-1}\right)e_{t-1}(\tilde{r}) = \frac{S_{t-1}}{S_{m-1}} = \left(\frac{N}{\ell-1}\right)e_{m}(\tilde{r}).
\]

\[
\frac{e_{t-1}(\tilde{r})}{e_{m-1}(\tilde{r})} \geq \left(\frac{N-m+1}{N-\ell+1}\right) \frac{e_{\ell}(\tilde{r})}{e_{m}(\tilde{r})} \geq \frac{e_{\ell}(\tilde{r})}{e_{m}(\tilde{r})}.
\]

\[\Box\]

**Lemma 2** For any positive vector \(\tilde{r} = (\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N) > 0\) and integers \((m, \ell)\) satisfying \(0 \leq m \leq \ell < N\), the inequality \(\frac{e_{\ell}(\tilde{r})}{e_{m}(\tilde{r})} \geq \frac{e_{\ell}(\tilde{r})}{e_{m}(\tilde{r})}\) holds, where \(\tilde{r}_{-1} = (\tilde{r}_2, \ldots, \tilde{r}_N)\).

**Proof.** When \(m = 0\), it is obvious from the positivity of \(\tilde{r}\). Let us consider cases that \(m > 0\). If we apply Lemma 1 to the positive vector \(\tilde{r}_{-1}\), then we obtain an inequality \(\frac{e_{t-1}(\tilde{r}_{-1})}{e_{m-1}(\tilde{r}_{-1})} \geq \frac{e_{t}(\tilde{r}_{-1})}{e_{m}(\tilde{r}_{-1})}\), which directly implies that

\[
\frac{e_{\ell}(\tilde{r})}{e_{m}(\tilde{r})} = \frac{\tilde{r}_1 e_{\ell-1}(\tilde{r}_{-1}) + e_{\ell}(\tilde{r}_{-1})}{\tilde{r}_1 e_{m-1}(\tilde{r}_{-1}) + e_{m}(\tilde{r}_{-1})} \geq \frac{e_{\ell}(\tilde{r}_{-1})}{e_{m}(\tilde{r}_{-1})}.
\]

\[\Box\]

### 3 Threshold Strategy

We deal with the sequence of independent 0/1 random variables \(X_1, X_2, \ldots, X_N\), where \(N\) is a given positive integer and the distribution is \(\Pr[X_i = 1] = p_i, \ Pr[X_i = 0] = 1 - p_i = q_i, 0 < p_i < 1\) for each \(i\). We define \(r_i = p_i/q_i\) for each \(i\). The \(r_i\)’s are called *odds*. Given a pair of integers \((k, \ell)\) satisfying \(1 \leq k \leq \ell < N\), we discuss a problem to predict the \(m\)-th last success satisfying \(k \leq m \leq \ell\), if any, with maximum probability at the time of its occurrence.
In the rest of this section, we denote the subvector \((r_i, r_{i+1}, \ldots, r_N)\) by \(r^{[i]}\) and \(\Pr[k \leq X_i + \cdots + X_N \leq \ell]\) by \(P^{[i]}\), i.e.,

\[
P^{[i]} \overset{\text{def}}{=} \Pr[k \leq X_i + \cdots + X_N \leq \ell] = \sum_{m=k}^{\ell} e_m(r^{[i]}) \prod_{j=i}^{N} (1 + r_j).
\]

We define an index \(i^*_s\) by

\[
i^*_s \overset{\text{def}}{=} \min \left\{ i \left| 1 \leq i \leq N - \ell \quad \text{and} \quad 1 > \frac{e_k(r^{[i+1]})}{e_{k-1}(r^{[i+1]})} \right. \right\}. \tag{2}
\]

When the minimum in (2) is taken on the empty set, we put \(i^*_s \overset{\text{def}}{=} N - \ell + 1\). Then, we have the following Lemma.

Lemma 3 \textit{The definition of } \(i^*_s\ \textit{directly implies the followings;}

\[
p^{[1]} \leq p^{[2]} \leq \cdots \leq p^{[i^*_s - 1]} \leq p^{[i^*_s]} > p^{[i^*_s + 1]} > \cdots > p^{[N-\ell+1]}, \tag{3}
\]

\[
\Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - p^{[i^*_s + 1]} \leq 0 \quad (1 \leq \forall i \leq i^*_s - 1), \tag{4}
\]

\[
\Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - p^{[i^*_s + 1]} > 0 \quad (i^*_s \leq \forall i \leq N - \ell). \tag{5}
\]

Proof. \textit{The definition of } \(i^*_s\ \textit{and Lemma 2 directly induce the followings;}

\[
\frac{e_k(r^{[1]})}{e_{k-1}(r^{[1]})} \geq \frac{e_k(r^{[2]})}{e_{k-1}(r^{[2]})} \geq \cdots \geq \frac{e_k(r^{[i^*_s]})}{e_{k-1}(r^{[i^*_s]})} \geq 1 \quad \text{and}
\]

\[
1 > \frac{e_k(r^{[i^*_s + 1]})}{e_{k-1}(r^{[i^*_s + 1]})} \geq \frac{e_k(r^{[i^*_s + 2]})}{e_{k-1}(r^{[i^*_s + 2]})} \geq \cdots \geq \frac{e_k(r^{[N-\ell+1]})}{e_{k-1}(r^{[N-\ell+1]})}.
\]

Thus, we have the inequalities

\[
e_{k-1}(r^{[i^*_s + 1]}) - e_k(r^{[i^*_s + 1]}) \begin{cases} 
0 \leq i^*_s - 1, & \text{if } 0 \leq \forall i \leq i^*_s - 1, \\
> 0 & \text{if } i^*_s \leq \forall i \leq N - \ell.
\end{cases}
\]

6
For any index \( i \in \{1, 2, \ldots, N - \ell \} \), the above inequalities imply that

\[
P[i] - P[i+1] = \frac{\sum_{m=k}^{\ell} e_m(r[i])}{(1 + r_i)(1 + r_{i+1}) \cdots (1 + r_N)} - P[i+1]
\]

\[
= \frac{\sum_{m=k}^{\ell} \left( e_m(r[i+1]) + r_i e_{m-1}(r[i+1]) \right)}{(1 + r_i)(1 + r_{i+1}) \cdots (1 + r_N)} - P[i+1]
\]

\[
r_i \left( e_{k-1}(r[i+1]) - e_\ell(r[i+1]) \right) + (1 + r_i) \sum_{m=k}^{\ell} e_m(r[i+1])
\]

\[
= \frac{r_i \left( e_{k-1}(r[i+1]) - e_\ell(r[i+1]) \right)}{(1 + r_i) \cdots (1 + r_N)} \left\{ \begin{array}{ll} 
< 0 & (0 \leq \forall i \leq i_* - 1), \\
> 0 & (i_* \leq \forall i \leq N - \ell), 
\end{array} \right.
\]

and thus, inequalities (3) are obtained.

For the remained statements, we have that

\[
\Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] = P[i+1] - P[i] - P[i+1]
\]

\[
= \frac{\sum_{m=k}^{\ell-1} e_m(r[i+1])}{\prod_{j=1}^{\ell-1} (1 + r_j)} - \frac{\sum_{m=k}^{\ell} e_m(r[i+1])}{\prod_{j=1}^{\ell} (1 + r_j)}
\]

\[
= \frac{e_{k-1}(r[i+1]) - e_\ell(r[i+1])}{\prod_{j=i}^{\ell} (1 + r_j)} \left\{ \begin{array}{ll} 
< 0 & (0 \leq \forall i \leq i_* - 1), \\
> 0 & (i_* \leq \forall i \leq N - \ell). 
\end{array} \right.
\]

Now we give an optimal rule.

**Theorem 2** An optimal rule is obtained by stopping at the first success \( X_i = 1 \) with \( i \geq i_* \).

**Proof.** Let \( P^{[i]}_{\text{win}} \) be the probability of win under an optimal stopping rule, if we start at the time instance just before observing random variable \( X_i \). First, we show that

\[
P^{[i]}_{\text{win}} = \left\{ \begin{array}{ll} 
P^{[i_*]} ( & \text{if } 1 \leq i \leq i_* - 1), \\
P^{[i]} ( & \text{if } i_* \leq i \leq N - \ell + 1). 
\end{array} \right.
\]
(i) Let us consider the case that \( i = N - \ell + 1 \). If we start with the observation of random variable \( X_{N - \ell + 1} \), the optimal strategy selects the first success and the probability of win is equal to

\[
P_{\text{win}}^{[N - \ell + 1]} = \frac{\sum_{m=k}^{\ell} e_m(r^{[N - \ell + 1]})}{\prod_{j=N - \ell + 1}^N r_j} = P_{\text{win}}^{[N - \ell + 1]}.
\]

Next, we show (6) for each \( i \in \{1, 2, \ldots, N - \ell \} \) by induction. The backward induction implies the following recursive formula;

\[
P_{\text{win}}^{[i+1]} = \begin{cases} 
P_{\text{win}}^{[i+1]} & (\text{if } \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - P_{\text{win}}^{[i+1]} \leq 0), \\
p_i \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] + q_i P_{\text{win}}^{[i+1]} & (\text{otherwise}).
\end{cases}
\]

(ii) Let \( i + 1 \) be an index satisfying \( i_{*} + 1 \leq i + 1 \leq N - \ell + 1 \) and \( P_{\text{win}}^{[i+1]} = P_{\text{win}}^{[i+1]} \).

Since \( i_{*} \leq i \leq N - \ell \), Lemma 3 (5) implies that

\[
0 < \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - P_{\text{win}}^{[i+1]} = \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - P_{\text{win}}^{[i+1]}.
\]

The above recursive formula shows that

\[
P_{\text{win}}^{[i]} = p_i \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] + q_i P_{\text{win}}^{[i+1]}
= p_i \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] + q_i P_{\text{win}}^{[i+1]}
= \sum_{m=k}^{\ell - 1} e_m(r^{[i+1]}) + \sum_{m=k}^{\ell} e_m(r^{[i+1]})
= \frac{p_i}{\prod_{j=i+1}^N (1 + r_j)} + \frac{q_i}{\prod_{j=i+1}^N (1 + r_j)}
= \frac{\sum_{m=k}^{\ell} (r_i e_{m-1}(r^{[i+1]}) + e_m(r^{[i+1]}))}{\prod_{j=i+1}^N (1 + r_j)}
= \frac{\sum_{m=k}^{\ell} e_m(r^{[i]})}{\prod_{j=i+1}^N (1 + r_j)} = P_{\text{win}}^{[i]}.
\]

(iii) Let \( i + 1 \) be an index satisfying \( 2 \leq i + 1 \leq i_{*} \) and \( P_{\text{win}}^{[i+1]} = P_{\text{win}}^{[i+1]} \). Since \( 1 \leq i \leq i_{*} - 1 \), Lemma 3 (3) and (4) imply that

\[
\Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - P_{\text{win}}^{[i+1]}
= \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - P_{\text{win}}^{[i+1]}
\leq \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] - P_{\text{win}}^{[i+1]} \leq 0.
\]
From the above recursive formula, we have that \( P_{\text{win}}^{[i]} = P_{\text{win}}^{[i+1]} = P^{[i^*]} \).

Now, we obtained (6). The above discussion directly implies that the optimal rule does not select any index less than \( i^* \) and selects the first variable \( X_i = 1 \) satisfying \( i^* \).

### 4 Lower Bounds

When we employ the optimal threshold strategy defined by (2), Theorem 2 says that the corresponding probability of win is equal to

\[
q_i q_{i+1} \cdots q_N \sum_{m=k}^{\ell} e_m(r_i, r_{i+1}, \ldots, r_N) = \frac{\sum_{m=k}^{\ell} e_m(r_i, r_{i+1}, \ldots, r_N)}{\prod_{j=i^*}^{\ell} (1 + r_j)}.
\]

In the rest of this section, we discuss a lower bound of the probability of win under the optimal stopping rule.

Newton’s inequalities directly implies the following lemma.

**Lemma 4** Every positive vector \( \bar{r} = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N) > 0 \) satisfies that

\[
S_{k-1}(\bar{r})^{\ell-m} \geq S_m(\bar{r})^{\ell-k+1} S_k(\bar{r})^{k-1-m}, \quad \forall m \in \{0, 1, 2, \ldots, k-1\}, \quad (7)
\]

\[
S_m(\bar{r})^{\ell-k+1} \geq S_{k-1}(\bar{r})^{\ell-m} S_k(\bar{r})^{m-k+1}, \quad \forall m \in \{k, k+1, \ldots, \ell\}, \quad (8)
\]

\[
S_\ell(\bar{r})^{m-k+1} \geq S_{k-1}(\bar{r})^{\ell-\ell} S_m(\bar{r})^{\ell-k+1}, \quad \forall m \in \{\ell+1, \ell+2, \ldots, N\}. \quad (9)
\]

**Proof.** The concavity of the sequence \((\log(S_0), \ldots, \log(S_N))\) implies that

\[
\log(S_{k-1}) \geq \frac{(\ell - k + 1) \log(S_m) + (k - 1 - m) \log(S_k)}{\ell - m}, \quad \forall m \in \{0, 1, \ldots, k-1\},
\]

\[
\log(S_m) \geq \frac{(\ell - m) \log(S_{k-1}) + (m - k + 1) \log(S_\ell)}{\ell - k + 1}, \quad \forall m \in \{k, k+1, \ldots, \ell\},
\]

\[
\log(S_\ell) \geq \frac{(m - \ell) \log(S_{k-1}) + (\ell - k + 1) \log(S_m)}{m - k + 1}, \quad \forall m \in \{\ell+1, \ell+2, \ldots, N\},
\]

and consequently inequalities (7) (8) and (9) are obtained. \( \square \)

**Theorem 3** Let us consider the problem of stopping at \( m \)-th last success with \( k \leq m \leq \ell \) defined on \( X_1, X_2, \ldots, X_N \) satisfying (1) \( r_i > 0 \) (\( \forall i \)), (2) \( 1 \leq k \leq \ell < N \), (3) \( 1 > \frac{e_k(\bar{r})}{e_{k-1}(\bar{r})} \) where \( \bar{r} = (r_{N-\ell+1}, r_{N-\ell+2}, \ldots, r_N) \in \mathbb{R}^{\ell} \)
and (4) \( \frac{\ell}{\ell - 1} \geq 1 \). Under the optimal stopping rule, the greatest lower bound of the probability of win is equal to

\[
\sum_{m=k}^{\ell} \left( \frac{N}{m} \right) \theta^m \left( 1 + \theta \right)^{N-m} = \frac{1}{r_{k+1}}.
\]

where \( \theta = \left( \frac{\binom{N}{k-1}}{\binom{N}{\ell}} \right)^{1} \).

**Proof.** Since the optimal stopping rule defined by (2) is an threshold strategy, the truncation of the subsequence \( X_1, X_2, \ldots, X_{i-1} \) does not affect the probability of win. Thus, we only need to consider a case where

\[
e_{k-1}(r_1, r_2, r_3, \ldots, r_N) - e_{\ell}(r_1, r_2, r_3, \ldots, r_N) > 0 \quad \text{and} \quad \left( e_{k-1}(r_1, r_2, r_3, \ldots, r_N) - e_{\ell}(r_1, r_2, r_3, \ldots, r_N) \right) \leq 0.
\]

Under assumptions (10) and (11), the optimal stopping rule is obtained by setting \( i_\ast = 1 \), and the probability of win is equal to

\[
V_N \overset{\text{def.}}{=} \sum_{m=k}^{\ell} e_m(r) \left( 1 + r_1 \right) \left( 1 + r_2 \right) \cdots \left( 1 + r_N \right).
\]

Thus, the greatest lower bound of the probability of win under the optimal stopping rule is equal to the optimal value of an optimization problem:

\[
P_1 \quad \min \quad V_N \overset{\text{def.}}{=} \sum_{m=k}^{\ell} e_m(r) \left( 1 + r_1 \right) \left( 1 + r_2 \right) \cdots \left( 1 + r_N \right)
\]

s. t. \( 0 < r_i \) (\( \forall i \in \{1, 2, \ldots, N\} \)),

\[
e_{k-1}(r) - e_{\ell}(r) > 0,
\]

\[
e_{k-1}(r) - e_{\ell}(r) \leq 0.
\]

where \( r_{-1} = (r_2, r_3, \ldots, r_N) \).

We show that we only need to consider feasible solutions satisfying constraint (12) by equality. Let \( r' \) be a feasible solution of P1 satisfying \( e_{k-1}(r') - e_{\ell}(r') < 0 \). We introduce a function \( f(r) : [0, r_1] \to \mathbb{R} \) defined by

\[
f(r) = e_{k-1}(r, r', r_3, \ldots, r_N) - e_{\ell}(r, r', r_3, \ldots, r_N),
\]

\[
10
\]
which is obtained by fixing $N - 1$ variables $\{r'_2, r'_3, \ldots, r'_N\}$. The assumption on $r'$ directly implies that

$$f(r'_1) = e_{k-1}(r') - e_\ell(r') < 0 < e_{k-1}(r'_{-1}) - e_\ell(r'_{-1}) = f(0).$$

From the continuity of $f(r)$, the mean-value theorem implies the existence of a value $r'' \in (0, r'_1)$ satisfying $f(r'') = 0$. Obviously, $(r'', r'_2, r'_3, \ldots, r'_N)$ is feasible to P1. The objective function value $V'_N$ corresponding to $r'$ becomes

$$V'_N = \sum_{m=k}^\ell e_m(r') \frac{(1 + r'_1)(1 + r'_2) \cdots (1 + r'_N)}{(1 + r'_1)(1 + r'_2) \cdots (1 + r'_N)} = \sum_{m=k}^\ell \left( e_m(r'_{-1}) - e_m(r'_{-1}) + (1 + r'_1)e_{m-1}(r'_{-1}) \right) \frac{(1 + r'_1)(1 + r'_2) \cdots (1 + r'_N)}{(1 + r'_1)(1 + r'_2) \cdots (1 + r'_N)}$$

$$= \frac{(-1)e_{k-1}(r'_{-1}) - e_\ell(r'_{-1})}{1 + r'_1} + \sum_{m=k}^\ell e_{m-1}(r'_{-1}).$$

Since $e_{k-1}(r'_{-1}) - e_\ell(r'_{-1}) > 0$ and $r'' \in (0, r'_1)$, the objective function value of $(r'', r'_2, r'_3, \ldots, r'_N)$ is strictly less than that of $r'$. As a result, we have shown that if a solution $r'$ feasible to P1 satisfies $e_{k-1}(r') - e_\ell(r') < 0$, then there exists a feasible solution $r''$ satisfying $e_{k-1}(r'') - e_\ell(r'') = 0$ with strictly smaller objective value. Thus, we only need to consider a set of feasible solutions of P1 satisfying $e_{k-1}(r') - e_\ell(r') = 0$.

Let $r^*$ be a feasible solution of P1 satisfying $e_{k-1}(r^*) - e_\ell(r^*) = 0$. Next, we derive an upper bound and/or a lower bound of $e_m(r^*)$. We introduce notations $\alpha = \left( \frac{S_k}{S_{k-1}} \right) ^{-\frac{1}{k+1}}$ and $\theta = \left( \frac{S_k}{S_{k-1}} \right) ^{-\frac{1}{k+1}}$, for simplicity. The equality $e_{k-1}(r^*) - e_\ell(r^*) = 0$ directly implies

$$\theta = \left( \frac{S_k}{S_{k-1}} \right) ^{-\frac{1}{k+1}} = \left( \frac{N}{k-1} \right) ^{\frac{1}{k+1}} e_\ell(r^*) = \left( \frac{N}{k-1} \right) ^{\frac{1}{k+1}}.$$
(i) Inequalities (7) imply that for any \( m \in \{0, 1, 2, \ldots, k - 1\} \),

\[
e_m(r^*) = \binom{N}{m} S_m \leq \binom{N}{m} \left( \frac{S_{k-1}^{k-m}}{S_{\ell-1-m}} \right)^{\frac{1}{\ell-k+1}} = \binom{N}{m} \alpha \theta^m.
\]

(ii) For each \( m \in \{k, k + 1, \ldots, \ell\} \), inequalities (8) give lower bounds (not upper bounds)

\[
e_m(r^*) = \binom{N}{m} S_m \geq \binom{N}{m} \left( \frac{S_{k-1}^{\ell-m} S_{\ell}^{m-k+1}}{S_{k-1}^{\ell-1}} \right)^{\frac{1}{\ell-k+1}} = \binom{N}{m} \alpha \theta^m.
\]

(iii) Inequalities (9) imply that for any \( m \in \{\ell + 1, \ell + 2, \ldots, N\} \),

\[
e_m(r^*) = \binom{N}{m} S_m \leq \binom{N}{m} \left( \frac{S_{\ell}^{m-\ell} S_{k-1}^{m-k+1}}{S_{k-1}^{m-\ell}} \right)^{\frac{1}{\ell-k+1}} = \binom{N}{m} \alpha \theta^m.
\]
Then the objective function value $V_N^*$ corresponding to $r^*$ satisfies that:

\[
\frac{1}{V_N^*} = \frac{(1 + r_1^*)(1 + r_2^*) \cdots (1 + r_N^*)}{\sum_{m=k}^\ell e_m(r^*)} = \sum_{m=0}^N e_m(r^*)
\]

\[
= \sum_{m=k}^{k-1} e_m(r^*) + \sum_{m=k}^\ell e_m(r^*) + \sum_{m=\ell+1}^N e_m(r^*)
\]

\[
\leq \sum_{m=k}^{k-1} \binom{N}{m} \alpha \theta^m + \sum_{m=\ell+1}^N \binom{N}{m} \alpha \theta^m = \sum_{m=k}^\ell \binom{N}{m} \theta^m + \sum_{m=\ell+1}^N \binom{N}{m} \theta^m
\]

and thus

\[
V_N^* \geq \frac{\sum_{m=k}^\ell \binom{N}{m} \theta^m}{(1 + \theta)^N}.
\]

Lastly, we discuss the tightness of the above lower bound. If we consider a case that $\tilde{r}_1 = \tilde{r}_2 = \cdots = \tilde{r}_N = \theta$, then we have that

\[
e_{k-1}(\tilde{r}) - e_\ell(\tilde{r}) = \left( \binom{N}{k-1} \right) \theta^{k-1} - \left( \binom{N}{\ell} \right) \theta^\ell
\]

\[
= \left( \binom{N}{k-1} \right) \left( \frac{N}{\ell} \right)^{k-1} \theta^{k-1} - \left( \binom{N}{\ell} \right) \left( \frac{N}{k-1} \right)^{\ell} \theta^{\ell}
\]

\[
= \left( \binom{N}{k-1} \right) \left( \frac{1}{\binom{k-1}{\ell} \theta^{k-1}} \right)^{k-1} - \left( \binom{N}{k-1} \right) \left( \frac{\ell}{k-1} \theta^{\ell} \right)^{\ell} = 0.
\]
and
\[
e_{k-1}(\hat{r}_-1) - e_\ell(\hat{r}_-1) = \binom{N-1}{k-1} \theta^{k-1} - \binom{N-1}{\ell} \theta^\ell
\]
\[
= \left( \frac{N-k+1}{N} \right) \binom{N}{k-1} \theta^{k-1} - \left( \frac{N-\ell}{N} \right) \binom{N}{\ell} \theta^\ell
\]
\[
= \left( \frac{N-k+1}{N} \right) e_{k-1}(\hat{r}) - \left( \frac{N-\ell}{N} \right) e_\ell(\hat{r})
\]
\[
\geq \left( \frac{N-\ell+1}{N} \right) e_{k-1}(\hat{r}) - \left( \frac{N-\ell}{N} \right) e_\ell(\hat{r}) = \frac{1}{N} e_{k-1}(\hat{r}) > 0.
\]
Thus, \( \hat{r} \) is feasible to P1 and corresponding probability of win (under the optimal stopping rule) attains the lower bound appearing in the right-hand side of (13). From the above, \( \hat{r} \) is optimal to P1, which induces the tightness of our lower bound.

Finally, we consider an asymptotic lower bound that is independent of N. The greatest lower bound of the probability of win (under the optimal stopping rule) is non-increasing with respect to N. Thus we discuss a case that \( N \to \infty \) and present a general lower bound.

**Corollary 1** Under assumptions in Theorem 3, the probability of win is greater than
\[
\exp \left( - \left( \frac{\ell!}{(k-1)!} \right) \sum_{m=k}^{\ell} \frac{1}{m!} \left( \frac{\ell!}{(k-1)!} \right) \frac{m!}{\ell^{k+1}} \right).
\]

**Proof.** It is easy to see that
\[
\sum_{m=k}^{\ell} \binom{N}{m} \theta^m \frac{(1+\theta)^N}{(1+\theta)^N} \geq e^{-N\theta} \sum_{m=k}^{\ell} \binom{N}{m} \theta^m = \exp \left( - \left( \binom{N}{1} \binom{N}{m} \theta^m \right) \sum_{m=k}^{\ell} \binom{N}{m} \theta^m.\right.
\]

14
For each \( m \in \{0, 1, \ldots, N\} \), we have an asymptotic value of \( \binom{N}{m} \theta^m \) by

\[
\binom{N}{m} \theta^m = \binom{N}{m} \left( \frac{N}{k-1} \right) \frac{m}{r-k+1}
\]

\[
= \frac{N!}{(N-m)!m!} \frac{\ell!(N-\ell)!}{(k-1)!(N-k+1)!} \frac{m}{r-k+1}
\]

\[
= \frac{1}{m!} \left( \frac{\ell!}{(k-1)!} \right) \frac{N!}{(N-m)!N^m} \frac{(N-\ell)!N^{k+1}}{(N-k+1)!} \frac{m}{r-k+1}
\]

\[
= \frac{1}{m!} \left( \frac{\ell!}{(k-1)!} \right) \frac{1}{(1 - \frac{m}{N})} \cdot \frac{1}{(1 - \frac{k}{N})} \cdot \frac{1}{(1 - \frac{\ell-1}{N})} \frac{m}{r-k+1}
\]

\[
\to \frac{1}{m!} \left( \frac{\ell!}{(k-1)!} \right) \frac{m}{r-k+1}, \quad \text{as } N \to \infty.
\]

From the above discussion, we obtain the asymptotic lower bound

\[
\lim_{N \to \infty} \sum_{m=k}^{\ell} \binom{N}{m} \theta^m \geq \lim_{N \to \infty} \exp \left( -\binom{N}{1} \theta \right) \sum_{m=k}^{\ell} \binom{N}{m} \theta^m
\]

\[
= \exp \left( -\left( \frac{\ell!}{(k-1)!} \right) \right) \sum_{m=k}^{\ell} \frac{1}{m!} \left( \frac{\ell!}{(k-1)!} \right) \frac{m}{r-k+1}.
\]

\]
\( e^{-1} \) (if \( \ell = k = 1 \)), which is a well-known bound for secretary problem and a lower bound shown by Bruss [2] for odds problem,

\( \frac{\ell!}{(\ell!)^k} e^\ell \) (if \( \ell = k \geq 1 \)) shown by Bruss and Paindaveine [3] for secretary problem,

\[ \exp \left( -\frac{1}{\ell!} \right) \sum_{m=1}^{\ell} \frac{\ell!}{m!} \left( \frac{\tau}{m} \right) \] (if \( \ell \geq k = 1 \)) shown by Tamaki [7] for the secretary problem, and by Matsui and Ano [5] for the odds problem.

References


